Chapter 1

Poset, Lattices, and Boolean Lattices

1.1 Partially ordered sets

In this section we introduce the reader into the basic notion of partially ordered set according to the following definition.

Definition 1.1.1 A *partially ordered set* (poset, for short) is a structure $\langle \mathcal{P}, \leq \rangle$ where $\mathcal{P}$ is a non-empty set and $\leq$ is a *partial order relation* on $\mathcal{P}$, i.e., a binary relation such that for arbitrary $a, b, c \in \mathcal{P}$ the following conditions hold:

(or1)	$a \leq a$			(reflexive)
(or2)	$a \leq b$ and $b \leq a$	imply	$a = b$	(antisymmetric)
(or3)	$a \leq b$ and $b \leq c$	imply	$a \leq c$	(transitive)

If $a \leq b$ we say that a is less than or equal to b, or that b is greater than or equal to a, also written in this case as $b \geq a$. Some other terminology frequently adopted in this case are that a *precedes* or is *smaller* than b and that b *follows* or *dominates* or is *larger* than a.

Example 1.1.2 Let us denote by $\mathcal{P}(X)$ the power set of a set (also universe) X, i.e., the collection of all subsets of X. The usual set theoretic inclusion on pairs A, B of subsets of X defined as "$A \subseteq B$ iff every element of A is also an element of B," (formally, "$a \in A$ implies $a \in B$") is a partial order relation on $\mathcal{P}(X)$. Thus $\langle \mathcal{P}(X), \subseteq \rangle$ is a poset. ■

Example 1.1.3 Let 2^X denote the collection of all Boolean valued functionals defined on the set X, i.e., the collection of all mappings $\chi : X \mapsto \{0, 1\}$. For any pair of such mappings $\chi_1, \chi_2 \in 2^X$, the binary relation defined as follows:

$$\chi_1 \leq \chi_2 \quad \text{iff} \quad \forall\, x \in X,\ \chi_1(x) \leq \chi_2(x) \tag{1.1}$$

is a partial order relation for 2^X, also called the pointwise order relation. Thus $\langle 2^X, \leq \rangle$ is a poset. ■

Let us introduce some notions which can be derived in any poset structure. First of all in a poset $\langle \mathcal{P}, \leq \rangle$ we can distinguish a *strict* partial ordering, written $<$, which means "x precedes y and x does not coincide with y"; formally, $x < y$ iff $x \leq y$ and $x \neq y$. For example, if $\leq$ is the set inclusion $\subseteq$, then $A < B$ means that A is a proper subset of B (there exists an element b in B which does not belong to A), i.e., $A \subset B$. Moreover, we say that z *covers* y iff $y < z$ and $y \leq x \leq z$ implies either $x = y$ or $x = z$ ($x < y$ and there is no element between x and y).

It is possible to give a graphical representation of a poset with a finite number of elements representing the fact that x *covers* y by drawing y below x and connecting them by a line segment according to the following figure:

The global figure so obtained is called the *Hasse diagram* of the finite poset.

Example 1.1.4 The Hasse diagram of a poset with five elements is presented in figure 1.1.

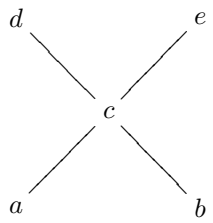

Figure 1.1: The poset S5

By the property of transitivity of a partial ordering, we read right from the diagram that $a \leq d$ since there exists the path $a \leq c$ and $c \leq d$ that moves steadily upward from a to d. Note that, in general, transitive relation can be run together without causing confusion: $x \leq y \leq z$ means $x \leq y$ and $y \leq z$, from which $x \leq z$. ■

Two elements a, b from a poset are called *comparable* iff one of the two cases, either $a \leq b$ or $b \leq a$, holds; two elements c, d which are not comparable are said to be *incomparable*. For example, in figure 1.1, the elements a and c are comparable, whereas a and b are not.

A poset $\mathcal{P}$ is a *totally* (or *linearly*) *ordered poset* iff $\forall\, x, y \in \mathcal{P}$ either $x \leq y$ or $y \leq x$, that is iff any two elements a, b from the poset $\mathcal{P}$ are comparable. A totally ordered poset is also called *chain*. Examples of totally ordered posets are the set of all natural numbers $\mathbb{N} = \{0, 1, 2, 3, 4, 5 \ldots\}$ with the standard ordering,

the set of all real numbers $\mathbb{R}$ and the unit interval $[0,1] := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ with the usual order on real numbers, this latter being a bounded chain.

It is possible to define special elements in a poset $\mathcal{P}$ and so to introduce operations in such a structure. First, an element $0 \in \mathcal{P}$ is said to be the *least* [Bir67] (or *minimum*, *first*, *zero*) element in $\mathcal{P}$ iff $0 \leq a$ for every $a \in \mathcal{P}$; if the least element 0 exists, then it is unique. Similarly, if a poset contains an element $1 \in \mathcal{P}$ such that $a \leq 1$ for every element a in the poset, then this element is uniquely determined and will be called the *greatest* [Bir67] (or *maximum*, *last*, *unit*) element. The poset S5 of figure 1.1 has neither least nor greatest element.

Definition 1.1.5 A partially ordered set with a minimum and a maximum element (also called *bounded poset*) is a structure $\langle \mathcal{P}, \leq, 0, 1 \rangle$, where $\leq$ is a partial order relation on $\mathcal{P}$ such that 0 and 1 are the *least* (*minimum*) and the *greatest* (*maximum*) element with respect to $\leq$, respectively. In other words, for all $b \in \mathcal{P}$: $0 \leq b \leq 1$.

In a poset with the least element $\langle \mathcal{P}, \leq, 0 \rangle$, every element that covers the 0 element is called *atom*. We say that $\mathcal{P}$ is *atomic* iff for every $x \neq 0$ at least one atom a in $\mathcal{P}$ exists such that $a \leq x$ holds.

Example 1.1.6 The power set $\mathcal{P}(X)$ of a universe X is atomic, every singlet set $\{a\}$ for $a \in X$ is an atom of the poset. For instance, in the lattice 2^2 of all subsets of the universe consisting of two elements $X = \{1, 2\}$ the singletons $\{1\}$ and $\{2\}$ are atoms:

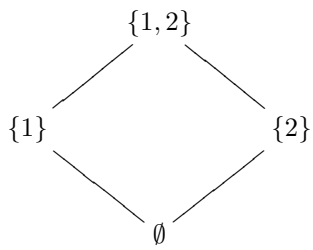

Figure 1.2: The atomic lattice 2^2 of all subsets of the universe $\{1, 2\}$.

■

Example 1.1.7 An example of poset without atoms is the real unit interval $[0, 1]$ endowed with the usual order on numbers. If to $[0, 1]$ we add an element $0 < a < 1$ such that a is incomparable with all elements in $(0, 1)$ (see the Hasse diagram of figure 1.3), we obtain a poset with an atom a which is non–atomic.

■

Let A be a subset of a poset $\langle \mathcal{P}, \leq \rangle$. Then the ordering $\leq$ of $\mathcal{P}$ induces an order on A in the following way: if $a, b \in A$, then $a \leq b$ as elements of A iff $a \leq b$ as elements of $\mathcal{P}$. In this case we shall say that the pair $\langle A, \leq \rangle$ is a subposet

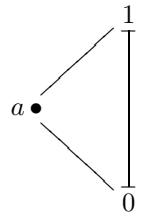

Figure 1.3: The non atomic real unit interval plus an atom a.

of $\langle \mathcal{P}, \leq \rangle$. Note that a subposet of a poset $\mathcal{P}$ may be totally ordered, also if $\mathcal{P}$ is not. For instance, in the non linear poset represented by the diagram of figure 1.1 the two subposets $\{b, c, e\}$ and $\{a, d\}$ are totally ordered, i.e, chains, whereas the subposet $\{a, b, c\}$ is not. Anyway, a subposet of a chain is always a chain.

Definition 1.1.8 Let $\mathcal{S} \neq \emptyset$ be a nonempty set, subset of the poset $\mathcal{P}$.

i) An element $x \in \mathcal{P}$ is said to be the *least upper bound* (l.u.b.) of $\mathcal{S}$, written $x = sup(\mathcal{S})$ (also $x = \vee\mathcal{S}$), iff:

   a) $\forall\, s \in \mathcal{S},\ s \leq x$;

   b) if $c \in \mathcal{P}$ satisfies $s \leq c$ for any $s \in \mathcal{S}$, then $x \leq c$.

   If only condition i) holds, the element s is said to be an *upper bound* of $\mathcal{S}$.

ii) An element $y \in \mathcal{P}$ is said to be the *greatest lower bound* (g.l.b.) of $\mathcal{S}$, written $y = inf(\mathcal{S})$ (also $y = \wedge\mathcal{S}$), iff:

   a) $\forall\, s \in \mathcal{S}\,; y \leq s$;

   b) if $d \in \mathcal{P}$ satisfies $d \leq s$ for any $s \in \mathcal{S}$, then $d \leq y$.

   If only condition i) holds, the element s is said to be a *lower bound* of $\mathcal{S}$.

Hence, an upper bound lies 'above' every element of the set $\mathcal{S}$, whereas a lower bound lies 'below' each element with respect to the involved partial ordering. The l.u.b. and the g.l.b. are the 'smallest' upper bound and the 'largest' lower bound of the set, respectively. If $\mathcal{S}$ is a finite set, say $\mathcal{S} = \{a_1, \ldots, a_n\}$, it is customary to write (if they exist) $\wedge_{i=1}^{n} a_i$ or $a_1 \wedge \ldots \wedge a_n$ and $\vee_{i=1}^{n} a_i$ or $a_1 \vee \ldots \vee a_n$ instead of $\wedge\mathcal{S}$ and $\vee\mathcal{S}$. In particular we denote by $a \vee b$ and $a \wedge b$, if they exist, the l.u.b. and the g.l.b. of the pair $a, b \in \mathcal{P}$, respectively. Let us note that in the case of a bounded poset $\mathcal{P}$ we have that $x \vee 0 = x$, $x \wedge 0 = 0$ and $x \wedge 1 = x$, $x \wedge 1 = x$ for any element $x \in \mathcal{P}$.

Example 1.1.9 In figure 1.1 the set $\{d, e\}$ has no l.u.b., whereas its g.l.b. is $c = d \wedge e$; the set $\{a, b\}$ has g.l.b. $c = a \vee b$, but the l.u.b. of this set does not exist. Moreover the whole poset has neither the minimum element 0 nor the maximum element 1. ■

Example 1.1.10 In the set of all integer numbers $\mathbb{Z}$, with the standard partial ordering, number 8 is an upper bound of the two subsets $S = \{-2, 0, 1, 3, 4\}$ and $Q = \{-4, -1, 2, 4, 8\}$, which is the l.u.b of Q but not of S. Similarly, -4 is a lower bound of both S and Q, which is the g.l.b of Q.

In the poset of all real numbers $\mathbb{R}$, number 8 is an upper bound of the open intervals $(-2, 6)$ and $(6, 8)$ and also of the closed intervals $[-5, -2]$ and $[-1, 8]$. Moreover, it is the l.u.b of both $(6, 8)$ and $[-1, 8]$; in this second case 8 belongs to the interval, contrary to the former one. In an analogous way, -5 is a lower bound of all these intervals, which is the g.l.b. of $[-5, -2]$.

There cannot be a number $\underline{\vee}$ (resp., $\overline{\wedge}$) that is an upper (resp., lower) bound of the whole set $\mathbb{Z}$ of all integer numbers, or of all real numbers $\mathbb{R}$ for that matter, for $\underline{\vee}$ (resp., $\overline{\wedge}$) is evidently strictly less (resp., great) than the number $\underline{\vee} + 1$ (resp., $\overline{\wedge} - 1$). ■

In the following proposition, for a pair of subsets Y_1 and Y_2 of a poset $\mathcal{P}$ we adopt the convention to denote by $Y_1 \leq Y_2$ the fact that $y_1 \leq y_2$ for every $y_1 \in Y_1$ and every $y_2 \in Y_2$.

Proposition 1.1.11 *Let $\langle \mathcal{P}, \leq \rangle$ be a poset.*

(1) Let Y be a subset of $\mathcal{P}$, then the elements $\sup(Y)$ *and* $\inf(Y)$*, if they exist are unique.*

(2) Let Y_1 and Y_2 be subsets of $\mathcal{P}$ such that $Y_1 \leq Y_2$ and both $\inf(Y_1)$ *and* $\sup(Y_2)$ *exist, then* $\inf(Y_1) \leq \sup(Y_2)$.

(3) $x \leq y$ iff $x \wedge y = x$ and $x \vee y = y$.

Proof – 1. Let $x_1 = \sup(Y)$ and $x_2 = \sup(Y)$. Then from (i-b) of definition 1.1.8 applied to x_1 we have in particular that $y \leq h$ for any $y \in Y$ implies $x_1 \leq h$. On the other hand, from (i-a) of the same definition applied to x_2 we have that $y \leq x_2$ for any $y \in Y$, hence $x_1 \leq x_2$. Reversing the role of these two elements we arrive to the result that $x_2 \leq x_1$ and so $x_1 = x_1$.

A similar proof can be given in the case of $\inf(Y)$. The proofs of points (2) and (3) are trivial. ■

Proposition 1.1.12 *A poset $\mathcal{P}$ which has the l.u.b. $\vee\mathcal{P}$ and the g.l.b. $\wedge\mathcal{P}$ is* bounded*: indeed, $\wedge\mathcal{P} \leq x \leq \vee\mathcal{P}$ for every $x \in \mathcal{P}$.*

Given the poset $\langle \mathcal{P}, \leq \rangle$, it is possible to construct the binary relation $\leq_c$ on $\mathcal{P}$ defined as $x \leq_c y$ iff $y \leq x$, which trivially is itself a partial ordering on $\mathcal{P}$ called the *converse* of $\leq$.

Other interesting notions involving posets are the following.

Definition 1.1.13 A *meet* (resp., *join*)–semilattice is a poset for which the g.l.b. $a \wedge b$ (resp., l.u.b. $a \vee b$) exists for any pair a, b of elements from the poset.

Example 1.1.14 Let us consider a finite set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, the *alphabet* of this example, whose elements a_i are symbols, also called *letters*. A *word* of the alphabet $\mathcal{A}$ is any finite sequence (also called *string*) of elements from $\mathcal{A}$, whose collection will be denoted by $\mathcal{A}^*$. The string with no letters at all, denoted by ϵ, is a special word of $\mathcal{A}^*$ called the *empty* or *null word*. The length of a word $w \in \mathcal{A}^*$, denoted by $|w|$, is the number of letters forming w, counting each appearance of a letter; in particular $|\epsilon| = 0$. For instance, in the case of a Boolean alphabet $\mathcal{A} = \{0, 1\}$ the finite sequence 0010101 is a word, i.e., an element from $\{0, 1\}^*$, of length 7.

A standard inner operation on $\mathcal{A}^*$ is the so-called *concatenation* (also *juxtaposition*): let $u = u_1u_2 \ldots u_h$ and $w = w_1w_2 \ldots w_k$ be two words in $\mathcal{A}^*$ (of length h and k respectively), then one can construct the new word $uw = u_1u_2 \ldots u_hw_1w_2 \ldots w_k$ which is another element of $\mathcal{A}^*$ (of length $h + k$). Concatenation with the empty word leaves the word unchanged: $w\epsilon = \epsilon w = w$ whatever be $w \in \mathcal{A}^*$. In the case of a nontrivial alphabet (i.e., containing two or more letters), this operation is not commutative: for instance in $\{0, 1\}^*$ the two words $u = 00$ and $w = 100$ are such that $uw = 00100$ and $wu = 10000$, which are different between them: $uw \neq wu$.

On $\mathcal{A}^*$ the binary relation $u \leq w$ means that the word $u \in \mathcal{A}^*$ is the initial segment of the word $w \in \mathcal{A}^*$, formally that there exists a word $v \in \mathcal{A}^*$ such that $uv = w$; in this case u is said to be a *prefix* of w. For instance in the case of the Boolean alphabet the two words 010 and 010011 are comparable, with $010 \leq 010011$, since there exists the word 011 such that $010011 = (010)(011)$. The poset $\{0, 1\}^*$ has the empty word ϵ as its minimum element: indeed, for any word $w \in \mathcal{A}^*$ the property $\epsilon w = w$ means that $\epsilon \leq w$. Moreover, it is a meet-semilattice: for instance, with respect to the meet we have $110 \wedge 10 = 1$, $010 \wedge 1 = \epsilon$, but the join of the words 10 and 01 does not exists since there is no upper bound w such that $10 \leq w$ and $01 \leq w$, i.e., two words u_1 and u_2 such that $10u_1 = 01u_2 = w$.

From the computational point of view, strings of an alphabet, for instance the Boolean one, "may be thought of as information encoded in binary form: the longer the string the greater the information content. Further, given any string v, we may think of elements u with $u < v$ as providing approximations to v." Extending in an obvious way these considerations to the collection $\mathcal{A}^{**}$ of all finite and infinite sequences of letters from the alphabet $\mathcal{A}$, "any infinite string is, in an [intuitive] sense, the limit of its finite initial substrings." [DP90] ■

1.2 Lattices

In the sequel, if they exist, we shall denote by $\vee a_j$ and $\wedge a_j$ the l.u.b. and the g.l.b. of any family $\{a_j : j \in J\}$ of elements from a poset $\mathcal{P}$, respectively.

Definition 1.2.1 A *lattice* is a poset $\langle \mathcal{L}, \leq \rangle$ such that both $x \wedge y := inf\{x, y\}$ (the lattice *meet* or *conjunction*) and $x \vee y := sup\{x, y\}$ (the lattice *join* or

disjunction) exist for every pair of elements $x, y \in \mathcal{L}$. To be precise, the lattice meet satisfies the conditions

$$\begin{array}{ll} \text{(LM1)} & a \wedge b \leq \{a, b\} \\ \text{(LM2)} & x \leq \{a, b\} \quad \text{implies} \quad x \leq a \wedge b \end{array}$$

whereas, the lattice join satisfies the conditions

$$\begin{array}{ll} \text{(LJ1)} & \{a, b\} \leq a \vee b \\ \text{(LJ2)} & \{a, b\} \leq x \quad \text{implies} \quad a \vee b \leq x \end{array}$$

Example 1.2.2 Let $\mathcal{P}(X)$ be the poset of all subsets of a given universe X with respect to the usual set theoretic inclusion relation $\subseteq$. A subset A of the universe X can be defined by a a property $P(x)$ characterizing just the elements of the universe which satisfies this property, and in this case we write $A = \{x \in X : P(x)\}$, where the symbol "$:$" has the meaning of *such that*. If the element x belongs to the set A, we write $x \in A$.

If the universe is finite, a set can be given by listing all its elements: $A = \{x_1, x_2, \ldots, x_n\}$. For instance, in the universe $[0, 10]_{\mathbb{N}}$ consisting of all integer numbers between 0 and 10, the set of even numbers less than 10 can be specified as $\{x : x \text{ is integer}, x < 10, x \mod 2 = 0\}$ or equivalently as $\{2, 4, 6, 8\}$.

A special set is the *empty set* which is a set with no elements, denoted by the symbol $\emptyset$. On the opposite site there is the *universe* X which is the set of all the elements of a given domain of interest. With respect to the poset $\langle \mathcal{P}(X), \subseteq \rangle$ of all subsets of a universe X, the set $\emptyset$ is the least element and X the greatest one:

$$\forall A \in \mathcal{P}(X) : \emptyset \subseteq A \subseteq X$$

Given two subsets $A = \{a \in X : P(a)\}$ and $B = \{b \in X : Q(b)\}$ of X defined by the properties P and Q respectively, their set intersection and union are given by:

$$\begin{aligned} A \cap B &= \{x \in X : P(x) \text{ and } Q(x)\} = \{x \in X : x \in A \text{ and } x \in B\} \\ A \cup B &= \{y \in X : P(y) \text{ or } Q(y)\} = \{y \in X : y \in A \text{ or } y \in B\} \end{aligned}$$

It is easy to verify that the poset $\langle \mathcal{P}(X), \subseteq, \emptyset, X \rangle$ is a lattice since $\inf\{A, B\}$ and $\sup\{A, B\}$ coincide with their set intersection and union, respectively:

$$A \wedge B = A \cap B \qquad \text{and} \qquad A \vee B = A \cup B. \tag{1.2}$$

Indeed, the set union $A \cup B$ contains both A and B, hence it is an upper bound of A and B. On the other hand, every subset C of X which contains both A and B (that is, every upper bound of A and B) contains their union too; thus $A \cup B$ is the smallest upper bound of A and B. We apply the same procedure to the intersection.

In order to stress the set operations of intersection $\cap$ and union $\cup$ as lattice meet and join respectively, the power set bounded lattice based on the universe X will be denoted in the sequel as the system $\langle \mathcal{P}(X), \cap, \cup, \emptyset, X \rangle$. ■

Remark 1.2.3 The formal definition of set union (resp., intersection) of two elements A and B of $\mathcal{P}(X)$ [as collection of all points from X which belong to at least (resp., both) the sets A and B] is very different from the formal definition of least upper (resp., greatest lower) bound of the same elements according to definition 1.1.8. We have just proved that these two formal definitions coincide on $\mathcal{P}(X)$ in the sense that they produce the same subsets of X.

Example 1.2.4 The poset 2^X of all Boolean valued functionals on X is a lattice where the meet and join of any pair of such functionals χ_1, χ_2 are defined for every point $x \in X$ as follows:

$$(\chi_1 \wedge \chi_2)(x) = \inf\{\chi_1(x), \chi_2(x)\} \text{ and } (\chi_1 \vee \chi_2)(x) = \sup\{\chi_1(x), \chi_2(x)\}. \tag{1.3}$$

■

Now, the following result is very important since it assures that the operators of meet $\wedge$ and join $\vee$ of a lattice $\mathcal{L}$ are interchangeable, i.e., it validates the dualization of all theorems with respect to $\mathcal{L}$ in the sense that the join may be replaced by the meet, and conversely.

Proposition 1.2.5 (Duality Principle) *Let us consider a lattice $\langle \mathcal{L}, \leq \rangle$, with associated converse lattice $\langle \mathcal{L}, \leq_c \rangle$. Then for any pair $x, y \in \mathcal{L}$ the following holds:*

(ia) If $x \wedge y$ is the lattice meet of the pair x, y with respect to the ordering $\leq$, then $x \wedge y$ is the lattice join of the same pair with respect to the converse *ordering $\leq_c$; formally, $x \vee_c y$ exists and $x \vee_c y = x \wedge y$.*

(ib) If $x \vee y$ is the lattice join of the pair x, y with respect to the ordering $\leq$, then $x \vee y$ is the lattice meet of the same pair with respect to the converse *ordering $\leq_c$; formally, $x \wedge_c y$ exists and $x \wedge_c y = x \vee y$.*

(iia) If $x \wedge_c y$ is the lattice meet of the pair x, y with respect to the ordering $\leq_c$, then $x \wedge_c y$ is the lattice join of the same pair with respect to the converse *ordering $\leq$; formally, $x \vee y$ exists and $x \vee y = x \wedge_c y$.*

(iib) If $x \vee_c y$ is the lattice join of the pair x, y with respect to the ordering $\leq_c$, then $x \vee_c y$ is the lattice meet of the same pair with respect to the converse *ordering $\leq$; formally, $x \wedge y$ exists and $x \wedge y = x \vee_c y$.*

The results (ia) and (iib) can be summarized in the unique formulation:

$$x \wedge y \ \textit{under} \ \leq \quad \textit{iff} \quad x \vee y \ \textit{under} \ \leq_c$$

Similarly, the results (ib) and (iia) can be summarized in the unique formulation:

$$x \vee y \ \textit{under} \ \leq \quad \textit{iff} \quad x \wedge y \ \textit{under} \ \leq_c$$

Proof – Let us set $h = x \wedge y$. Then the condition $h \leq \{x, y\}$ (lower bound of the pair x, y) translates in the "converse" condition $\{x, y\} \leq_c h$ (upper bound of the same pair). Moreover, the fact that $d \leq \{x, y\}$ implies $d \leq h$ (l.u.b. of x, y) leads to the fact that $\{x, y\} \leq_c d$ implies $h \leq_c d$. So we have proved that if $x \wedge y$ is the lattice meet of the pair x, y with respect to the ordering $\leq$, then the same element $x \wedge y$ is the lattice join of the same pair with respect to the ordering $\leq_c$; formally, $x \vee_c y$ exists and $x \vee_c y = x \wedge y$.

The case of the join (ib), and the corresponding dual, can be proved likewise. ■

Definition 1.2.6 A σ-lattice is a poset for which the sup and inf exist for any countable collection of its elements.

Example 1.2.7 Let us recall that a *measurable space* is a pair $(X, \mathcal{A}(X))$ consisting of a nonempty set X and a family $\mathcal{A}(X)$ of its subsets, called *measurable sets*, satisfying the following conditions:

(M1) The empty set is measurable: $\emptyset \in \mathcal{A}(x)$.

(M2) The set union of countable measurable sets is measurable: let $M_n \in \mathcal{A}(x) : n \in \mathbb{N}$, then $\bigcup_{n \in \mathbb{N}} M_n \in \mathcal{A}(x)$.

(M3) The set complement of a measurable set is measurable: $M^c \in \mathcal{A}(X)$ for every $M \in \mathcal{A}(X)$.

The σ-algebra $(\mathcal{A}(X), \subseteq)$ equipped with the set inclusion is a σ-lattice, bounded by the least element $\emptyset \in \mathcal{A}(X)$ and the greatest one $X = \emptyset^c \in \mathcal{A}(X)$. Indeed for any countable family of measurable sets, from (M2) $\bigvee M_n = \bigcup M_n$ follows and since $\bigcap M_n = (\bigcup (M_n)^c)^c \in \mathcal{A}(X)$ from (M2) and (M3) also $\bigwedge M_n = \bigcap M_n$. Note that in this example the meet is the set intersection and the join the set union. ■

Definition 1.2.8 A lattice $\mathcal{L}$ is *complete* iff both $\inf(S)$ and $\sup(S)$ exist for every nonempty subset, i.e., any $S \subseteq \mathcal{L}$ with $S \neq \emptyset$.

Every finite lattice is obviously complete. Every complete lattice $\mathcal{L}$ is bounded since $0 := \inf(\mathcal{L})$ is the least element and $1 := \sup(\mathcal{L})$ is the greatest one.

Example 1.2.9 Given a universe X, the two lattices $\mathcal{P}(X)$ (of all subsets of X) and 2^X (of all Boolean–valued functionals defined on X) are complete lattices since the operations introduced in equations (1.2) and (1.3) can be extended to arbitrary families. ■

Example 1.2.10 Let us recall that a *topological space* is a pair $(X, \mathcal{C}(X))$ consisting of a set X and a family $\mathcal{C}(X)$ of its subsets, called *closed sets*, satisfying the following conditions:

(T1) The empty set and the whole space X are closed $\emptyset, X \in \mathcal{C}(X)$.

(T2) The set intersection of *any* family of closed sets is closed: let $C_i \in \mathcal{C}(X)$, with i running on an arbitrary index set I, then $\bigcap_{i\in I} C_i \in \mathcal{C}(X)$.

(T3) The *finite* union of closed sets is closed: $\bigcup_{j=1,\dots,k} C_k \in \mathcal{C}(X)$ for any finite collection $C_1, \dots, C_k$ of closed sets.

Then $(\mathcal{C}(X), \subseteq)$, being $\subseteq$ the usual set inclusion, is a complete lattice in which the meet is the set intersection ($\wedge C_i = \cap_i C_i$), whereas the join is the closure of the set union ($\vee C_j = \overline{\cup_j C_j}$, obtained as the set intersection of all closed sets containing $\cup C_j$, formally $\vee C_j = \cap\{K \in \mathcal{C}(X) : \cup C_j \subseteq K\}$, which is closed owing to (T2)). Therefore, if we exclude the case of a finite number of closed sets, the join is in general different from the set union.

For instance, if one considers the real line $\mathbb{R}$ with the standard Euclidean topology, the sequence of closed sets $I_n = [-\frac{1}{n}, 1 + \frac{1}{n}]$ (for $n \in \mathbb{N}$) has the set union

$$\bigcup_{n\in\mathbb{N}} I_n = (0,1)$$

which is an open set whose closure is the interval $[0,1]$, the intersection of all closed sets containing the union $\cup I_n = (0,1)$:

$$\bigvee_{n\in\mathbb{N}} I_n = [0,1] \neq (0,1) = \bigcup_{n\in\mathbb{N}} I_n$$

■

This is not the unique situation in which some lattice operation does not coincide with the set theoretic ones.

Example 1.2.11 Let $\mathcal{V}(X) = \langle X, +, \cdot_{\mathbb{K}} \rangle$ be a *linear space*, i.e., a nonempty set of *vectors* containing a privileged element $\underline{0}$ and equipped with a binary operation of sum $+ : X \times X \to X$ (of abelian group with $\underline{0}$ as the zero vector) and an external operation $\cdot : \mathbb{K} \times X \to X$ of multiplication of *scalars* from a field $\mathbb{K}$ (usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) with vectors from X to produce a new vector in X.

A *linear subspace*, simply subspace in the sequel, is a nonempty subset M of X (in general $\underline{0} \in M$), which is closed with respect to the inner sum (for any pair of vectors $x, y \in M$ it is $x+y \in M$) and the external product of scalars with vectors ($\alpha \in \mathbb{K}$ and $x \in M$ imply $\alpha \cdot x \in M$). The collection of all subspaces of X will be denoted by $\mathcal{M}(X)$, which is a poset with respect to the set inclusion, bounded by the least element $\{\underline{0}\}$, the subspace consisting of the unique zero vector, and the greatest element X, the whole space: formally, for any subspace $M \in \mathcal{M}(X)$ it is $\{\underline{0}\} \subseteq M \subseteq X$ with both $\{0\}, X \in \mathcal{M}(X)$. Then it is easy to show that the set intersection of *any* family of subspaces is a subspace too:

$$\bigcap_{i\in I} M_i \in \mathcal{M}(X) \quad \text{for any family of subspaces } M_i \in \mathcal{M}(X)$$

In this way, the lattice meet of any family of subspaces is their set intersection:

$$\bigwedge_{i\in I} M_i = \bigcap_{i\in I} M_i$$

whereas, the lattice join of the same family is the subspace *generated* by the set union, which is the intersection of all subspaces containing the union $\cup M_i$:

$$\bigvee_{i\in I} M_i = \bigcap \{H \in \mathcal{M}(X) : \cup_{i\in I} M_i \subseteq H\}$$

For instance, in the linear space $\mathbb{R}^2$ the two subspaces $M_x = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $M_{\pi/4} = \{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ are both (one–dimensional) subspaces for which $M_x \wedge M_{\pi/4} = M_x \cap M_{\pi/4} = \{\underline{0}\}$ and $M_x \vee M_{\pi/4} = \mathbb{R}^2$ (since $\mathbb{R}^2$ is the unique subspace which contains both M_x and $M_{\pi/4}$), this latter being strictly greater than the set union of these two subspaces.

If one represents any one-dimensional subspace of $\mathbb{R}^2$ by the angle $\alpha \in [0,\pi)$ between this subspace and the x–axis, written as M_α, then the complete lattice $\mathcal{M}(\mathbb{R}^2)$ can be represented by the Hasse diagram of the right side of figure 1.4, where it is clear that for any pair of different one–dimensional subspaces M_α and M_β it is $M_\alpha \wedge M_\beta = \{\underline{0}\}$ and $M_\alpha \vee M_\beta = \mathbb{R}^2$.

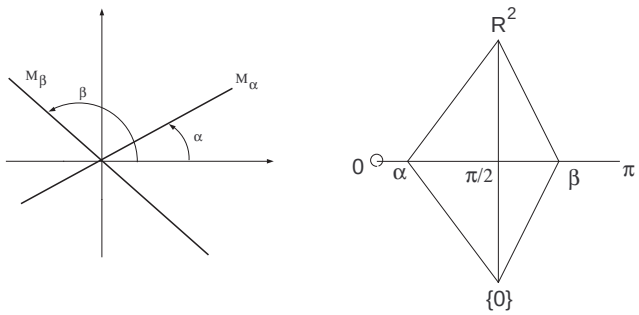

Figure 1.4: The 2 dimensional linear space $\mathbb{R}^2$ with two one–dimensional subspaces M_α, M_β (left) and the corresponding Hasse diagram representation of the (non distributive) lattice (right).

Note that in this lattice the distributivity conditions of the two lattice meet and join operations does not hold. For instance, if one considers two different one–dimensional subspaces $\alpha \neq \beta$, which are both different from the x–axis too, then $M_\alpha \wedge (M_\beta \vee M_x) = M_\alpha \wedge \mathbb{R}^2 = M_\alpha$, whereas $(M_\alpha \wedge M_\beta) \vee (M_\alpha \wedge M_x) = \{\underline{0}\} \vee \{\underline{0}\} = \{\underline{0}\}$. Similarly, $M_\alpha \vee (M_\beta \wedge M_x) = M_\alpha \vee \{\underline{0}\} = M_\alpha$, whereas $(M_\alpha \vee M_\beta) \wedge (M_\alpha \vee M_x) = \mathbb{R}^2 \wedge \mathbb{R}^2 = \mathbb{R}^2$. ■

1.2.1 Lattices and related algebras

As we have seen before, a partially ordered set $\langle \mathcal{L}, \leq \rangle$ is a lattice iff both the g.l.b. and the l.u.b exist for every pair of elements $x, y \in \mathcal{L}$ (and are unique according to point (1) of proposition 1.1.11 applied to the subset $Y = \{x, y\}$). Because of their properties of existence and uniqueness, we can think to $\inf(x, y)$ and $\sup(x, y)$, with x and y running on the lattice $\mathcal{L}$, as the results of two binary operations on $\mathcal{L}$. In this way a lattice can be considered as a particular algebraic structure $\langle \mathcal{L}, \wedge, \vee \rangle$, based on a set $\mathcal{L}$ equipped with two inner composition laws.

$$\wedge : \mathcal{L} \times \mathcal{L} \mapsto \mathcal{L},\ (x, y) \to x \wedge y := \inf\{x, y\} \tag{1.4}$$

$$\vee : \mathcal{L} \times \mathcal{L} \mapsto \mathcal{L},\ (x, y) \to x \vee y := \sup\{x, y\} \tag{1.5}$$

It is possible to make a link between the lattice $\langle \mathcal{L}, \leq \rangle$ and the algebra $\langle \mathcal{L}, \wedge, \vee \rangle$, whose proof can be found in [Bir67, p. 8].

Theorem 1.2.12 *Let $\langle \mathcal{L}, \leq \rangle$ be a lattice. Then the algebraic structure $\langle \mathcal{L}, \wedge, \vee \rangle$, where $\wedge$ and $\vee$ are the binary operations on $\mathcal{L}$ of meet* (1.4) *and join* (1.5), *satisfies the following laws for every $x, y, z \in \mathcal{L}$:*

$$\begin{array}{lll}
(L1) & x \wedge x = x\,, \quad \textit{and} \quad x \vee x = x & \text{(idempotence)} \\
(L2) & x \wedge y = y \wedge x \quad \textit{and} \quad x \vee y = y \vee x & \text{(commutativity)} \\
(L3) & x \wedge (y \wedge z) = (x \wedge y) \wedge z \quad \textit{and} \quad x \vee (y \vee z) = (x \vee y) \vee z & \text{(associativity)} \\
(L4) & x \wedge (x \vee y) = x \quad \textit{and} \quad x \vee (x \wedge y) = x & \text{(absorption)}
\end{array}$$

Moreover the two consistence properties hold:

$$\begin{array}{llll}
(L5a) & x \leq y & \textit{is equivalent to} & x = x \wedge y \\
(L5b) & x \leq y & \textit{is equivalent to} & y = x \vee y
\end{array}$$

Let us notice that according to this result the algebraic version $\langle \mathcal{L}, \wedge, \vee \rangle$ of the lattice is defined in an entirely *equational* way. Moreover, the peculiar form of the properties (L1)–(L4) gives a corroboration of the duality principle expressed in proposition 1.2.5 since if we consider one of the properties defining a lattice and we replace each symbol $\wedge$ by $\vee$ and each $\vee$ with $\wedge$, then we obtain the dual property. Thus it follows that any theorem, result, identity that holds for a lattice will remain true if we dualize it, that is, if we replace each $\wedge$ by $\vee$ and each $\vee$ by $\wedge$.

We recall the following result whose proof can be found in [Bir67, p.22]

Theorem 1.2.13 *Postulates (L2)–(L4) for lattices imply (L1), but the six identities of (L2)–(L4) are independent.*

Properties (L2)–(L4) characterize lattices completely, in a sense expressed by the following theorem.

Theorem 1.2.14 *Let $\langle \mathcal{L}, \wedge, \vee \rangle$ be an algebraic structure where:*

(i) $\mathcal{L}$ is a non empty set,

(ii) $\wedge$ *and* $\vee$ *are two binary operations on* $\mathcal{L}$ *satisfying properties (L2)-(L4).*

If we define the following binary relation on $\mathcal{L}$*:*

$$x \leq y \quad \textit{iff} \quad y = x \vee y \quad (\textit{equivalently, iff} \quad x = x \wedge y) \tag{1.6}$$

then $\langle \mathcal{L}, \leq \rangle$ *is a lattice such that*

$$x \vee y = sup\,\{x, y\} \quad \textit{and} \quad x \wedge y = inf\,\{x, y\}.$$

Moreover, the following result assures the "categorical equivalence" between the two versions of the lattice notions, the poset one and the algebraic one.

Proposition 1.2.15 *Starting with a poset version of a lattice* $\langle \mathcal{L}, \leq \rangle$*, forming its algebraic version* $\langle \mathcal{L}, \wedge, \vee \rangle$ *and then forming its corresponding poset version, denoted by* $\langle \mathcal{L}, \preceq \rangle$*, then this latter gives us back our original poset structure, i.e.,* $\preceq$ *is the partial order* $\leq$ *we started with.*

Vice versa, starting from an algebraic version of a lattice $\langle \mathcal{L}, \wedge, \vee \rangle$*, forming its poset version* $\langle \mathcal{L}, \leq \rangle$ *and then forming its corresponding algebraic version, denoted by* $\langle \mathcal{L}, \sqcap, \vee \rangle$*, then one obtains the original algebraic structure, i.e.,* $\wedge = \sqcap$ *and* $\vee = \sqcup$*.*

Until now we have considered inf and sup as two binary operations holding between two generic elements of a set. It is time for us to ask if we can find neutral elements for these operations.

Proposition 1.2.16 *Let* $\langle \mathcal{L}, \wedge, \vee \rangle$ *be the algebraic formulation of a lattice.*

i) If a neutral element exists for the binary operation $\vee$*, called the algebraic* zero*, then this element is unique and turns out to be the minimum element* 0 *with respect to partial ordering* (1.6) *induced from the algebra.*

ii) If the neutral element exists for the binary operation $\wedge$*, called the algebraic* unity*, then this element is unique and turns out to be the maximum element* 1 *with respect to partial ordering* (1.6) *induced from the algebra.*

Proof – By definition of $\vee$–neutral element we have that $\forall x \in \mathcal{L}$: $x \vee 0 = x$, that is $0 \leq x$ for every element x, according to (1.6). This result means that if the zero element exists in the algebraic version of a lattice $\langle \mathcal{L}, \wedge, \vee \rangle$, then it is the minimum element relatively to the induced poset $\langle \mathcal{L}, \leq \rangle$, and it is unique at the same time.

On the other side, always according to (1.6), $x \wedge 1 = x$ for every $x \in \mathcal{L}$ is equivalent to $x \leq 1$; and so if the algebraic unity 1 exists in $\mathcal{L}$, then this unity is unique and is the maximum element in the induced poset structure $\langle \mathcal{L}, \leq \rangle$. ■

The case of a bounded lattice has an interesting peculiar algebraic characterization, at least in terms of the join operation.

Proposition 1.2.17 *Let $\langle \mathcal{L}, \leq, 0\rangle$ be a bounded lattice, then the binary operation of join $\vee$ satisfies the equations for all x, y, z:*

$(L1)$	$x \vee x = x$	(idempotence)
$(L2)$	$x \vee y = y \vee x$	(commutativity)
$(L3)$	$x \vee (y \vee z) = (x \vee y) \vee z$	(associativity)
$(L0)$	$x \vee 0 = x$	(unit element)

Briefly the structure $\langle \mathcal{L}, \vee, 0\rangle$ is a commutative monoid *(semigroup with unity) in which any element is idempotent.*

Conversely we have that if $\langle \mathcal{L}, \vee, 0\rangle$ is a commutative monoid in which every element is idempotent, then there exists a unique partial ordering on $\mathcal{L}$ such that $x \vee y$ is the lattice join of any pair x and y, and 0 is the least element.

This means that this structure is a join–semilattice and so a semilattice can be defined either in terms of the order relation or in terms of join operation.

Definition 1.2.18 Two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are said to be *isomorphic* if there exists a bijective mapping of $\mathcal{L}_1$ into $\mathcal{L}_2$, say $\phi : \mathcal{L}_1 \mapsto \mathcal{L}_2$, such that for arbitrary $x, y \in \mathcal{L}_1$

$$\phi(x \wedge_1 y) = \phi(x) \wedge_2 \phi(y) \quad \text{and} \quad \phi(x \vee_1 y) = \phi(x) \vee_2 \phi(y) \tag{1.7}$$

Of course, if $\phi : \mathcal{L}_1 \mapsto \mathcal{L}_2$ is a lattice isomorphism, then its inverse $\phi^{-1} : \mathcal{L}_2 \mapsto \mathcal{L}_1$ is a lattice isomorphism too. Trivially, a lattice isomorphism preserves the ordering, i.e., it is a poset isomorphism:

$$\begin{aligned} x \leq_1 y \quad &\text{implies} \quad \phi(x) \leq_2 \phi(y) \\ \phi(x) \leq_2 \phi(y) \quad &\text{implies} \quad x \leq_1 y \end{aligned}$$

Indeed, $x \leq_1 y$ is equivalent to $x = x \wedge_1 y$ and so $\phi(x) = \phi(x) \wedge_2 \phi(y)$ which means that $\phi(x) \leq_2 \phi(y)$. The second condition can be proved in a similar, also if a little bit complicated, way.

Proposition 1.2.19 *If two bounded lattices are* isomorphic*, then the further conditions are satisfied:*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(1) = 1$$

Proof – If we apply the first equation of (1.7) to the particular case $y = 0$ we obtain that for every x it is $\phi(0) = \phi(x) \wedge \phi(0)$, i.e., $\phi(0) \leq \phi(x)$. Since this inequality holds for every element x of the lattice $\mathcal{L}_1$, let us denote by $x_0 = \phi^{-1}(0)$ the inverse imagine of the $0 \in \mathcal{L}_2$; then we have that $\phi(0) \leq 0$, i.e., $\phi(0) = 0$. The second statement can be analogously proved. ■

Example 1.2.20 Let us consider the two bounded lattices $\mathcal{P}(X)$ and 2^X of examples 1.1.2 and 1.1.3, respectively. The lattice structure of the power lattice

$\mathcal{P}(X)$ of the universe X has been widely discussed in example 1.2.2. Now, let us consider the mapping

$$\chi : \mathcal{P}(X) \mapsto 2^X$$

associating with any subset A of X its characteristic functional $\chi_A : X \mapsto \{0, 1\}$, i.e., the Boolean valued functional defined as $\chi_A(x) = 1$ if $x \in A$, and $= 0$ otherwise.

This mapping is a bijection since (1) it is injective: $A \neq B$ trivially implies that $\chi_A \neq \chi_B$; and (2) it is surjective since for any $\psi : X \mapsto \{0, 1\}$ in 2^X, constructed the subset $A(\psi) := \{x \in X : \psi(x) = 1\}$, the corresponding characteristic functional is such that $\chi_{A(\psi)} = \psi$.

With respect to the meet and join operations, recalling that the lattice meet and join of two characteristic functionals $\chi_A, \chi_B \in 2^X$ with respect to the partial ordering (1.1) both exist (see the example 1.2.4), we have the following correspondences whatever be $A, B \in \mathcal{P}(X)$:

$$A \cap B \xrightarrow{\chi} \chi_{A\cap B} = \chi_A \wedge \chi_B$$
$$A \cup B \xrightarrow{\chi} \chi_{A\cup B} = \chi_A \vee \chi_B$$

i.e., χ is a lattice isomorphism, which allows one to identify the subset A of the universe X with its characteristic functional χ_A. ■

1.3 Distributive and Modular Lattices

In the context of lattices we can distinguish at least two different classes: *distributive lattices* and *modular lattices*.

Definition 1.3.1 A lattice $\mathcal{L}$ is said to be *distributive* if and only if the following equalities hold :

(d1) for every $x, y, z \in \mathcal{L}$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,

(d2) for every $x, y, z \in \mathcal{L}$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

> **Remark 1.3.2** As remarked in [Bir67, p. 11] conditions (d1) and (d2) are mutually equivalent, and so only one of these equalities is enough to define a distributive lattice [either (d1) or (d2)]. Notice that in the proof of this equivalence property a determinant role is played by the universal quantifier in the sense that these two conditions are equivalent if they hold for *any* triple of elements from the lattice ($\forall x, y, z \in \mathcal{L}$).
>
> If the lattice is not distributive, there might be particular triples of individual elements ($\exists x_0, y_0, z_0 \in \mathcal{L}$) such that $x_0 \vee (y_0 \wedge z_0) = (x_0 \vee y_0) \wedge (x_0 \vee z_0)$, but with respect to which $x_0 \wedge (y_0 \vee z_0) \neq (x_0 \wedge y_)) \vee (x_0 \wedge z_0)$, and vice versa.

Example 1.3.3 The lattice N5 shown in figure 1.5 is not distributive.
Indeed, one has that $b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = b$, i.e., the (ordered) triple (b, a, c) is (d2)–distributive, but $b \vee (a \wedge c) = b \neq a = (b \vee a) \wedge (b \vee c)$, i.e., this triple is not (d1)–distributive. ■

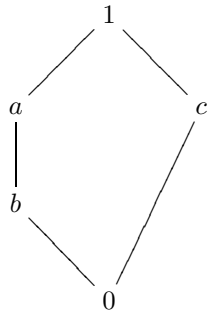

Figure 1.5: The non distributive lattice N5.

In [Sho51] one can found the following result which is the distributive analogous of theorem 1.2.14 linking lattices and algebraic structures.

Theorem 1.3.4 *Let $\langle \mathcal{L}, \wedge, \vee \rangle$ be an algebraic structure where:*

(1) $\mathcal{L}$ is a non empty set,

(2) $\wedge$ and $\vee$ are two binary operations on $\mathcal{L}$ satisfying properties:

(DL1) $x = x \wedge (x \vee y)$
(DL2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$

Then, $\mathcal{L}$ is a distributive lattice.

Definition 1.3.5 A lattice $\mathcal{L}$ is said to be *modular* if and only if the following equality holds:

(m1) for every $x, y, z \in \mathcal{L}$, $\quad x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Remark 1.3.6 As (d1) and (d2) are dual forms, one is suggested to consider the dual of (m1) whose form is

(m2) $\quad z \leq x \quad$ implies $\quad x \wedge (y \vee z) = (x \wedge y) \vee z$.

but if one put the substitutions $\hat{x} := z$ and $\hat{z} := x$ the (m2) assumes the form

(m2) $\quad \hat{x} \leq \hat{z} \quad$ implies $\quad \hat{z} \wedge (y \vee \hat{x}) = (\hat{z} \wedge y) \vee \hat{x}$.

which is nothing else than the original (m1).

From definitions 1.3.1 and 1.3.5 it is straightforward to prove that every distributive lattice is modular too: indeed, from (d1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ the condition $x \leq z$, i.e., $z = x \vee z$, leads to (m1). The vice versa is in general not true since it is possible to find modular lattices which are not distributive.

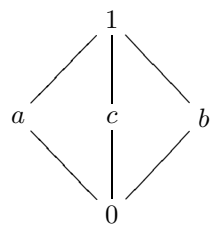

Figure 1.6: The modular lattice M5.

Example 1.3.7 This is exactly the case of the finite lattice M5 whose Hasse diagram is represented in figure 1.6.

Indeed we have $a \wedge (b \vee c) = a \wedge 1 = a$ and $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$, from which it follows that $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$, i.e., this lattice is not distributive.

We prove now that this lattice is modular. Let $x \leq z$ be two nontrivial ($\neq 0, 1$) elements in M5, then from the particular structure of M5 it follows that $x \vee (y \wedge z) = x \vee (y \wedge x) = x$ and $(x \vee y) \wedge z = (x \vee y) \wedge x = x$, by absorption (this latter does not hold in the lattice of 1.5 where for instance $b \leq a$, but $(b \vee c) \wedge a \neq (b \vee c) \wedge b$). In the non trivial cases we have to consider two subcases $0 < a$ and $a < 1$ by symmetry; in the first we obtain $0 \vee (y \wedge a) = y \wedge a$ and $(0 \vee y) \wedge a = y \wedge a$, while in the latter we have $a \vee (y \wedge 1) = a \vee y$ and $(a \vee y) \wedge 1 = a \vee y$.

As a concrete example of this behavior let us consider the complete lattice $\mathcal{M}(\mathbb{R}^2)$ of all subspaces of the two dimensional linear space $\mathbb{R}^2$. Then if one set $0 := \{\underline{0}\}$ (the subspace consisting of the zero vector) and $1 = \mathbb{R}^2$, the following subspaces realize the Hasse diagram of figure 1.6:

$$a := M_x = \{(x, 0) : x \in \mathbb{R}\} \qquad b := M_y = \{(0, y) : y \in \mathbb{R}\}$$
$$c := M_{\pi/4} = \{(x, x) : x \in \mathbb{R}\}$$

■

In general it can be proved that

> any *finite dimensional* (real or complex) linear space is such that the complete lattice of all its subspaces is modular: "In particular, set–products and straight linear sums are known to satisfy the so-called modular identity." [BvN36].

This result is very simple to prove in the two dimensional case, whereas the proof of the general finite dimensional case can be found in [BvN36, footnote 23 of section 11].

Example 1.3.8 The lattice $\mathcal{M}(\mathbb{R}^2)$ of all subspaces of the real vector space $\mathbb{R}^2$ has been represented by the Hasse diagram in the right side of figure 1.4. In

order to verify the condition (m1) of definition 1.3.5, we first consider the trivial case of $\alpha = \gamma$. Since for any β, it is $\alpha \leq \beta \vee \alpha$, we have $\alpha = \alpha \wedge (\beta \vee \alpha)$. If β represents any subspace different from α and $\mathbb{R}^2$, then $\beta \wedge \alpha = \{\underline{0}\}$ and so $\alpha = \alpha \vee \{\underline{0}\} = \alpha \vee (\beta \wedge \alpha)$; if $\beta = \mathbb{R}^2$, then $\alpha = \mathbb{R}^2 \wedge \alpha$ and so $\alpha = \alpha \vee \alpha = \alpha \vee (\mathbb{R}^2 \wedge \alpha)$. Thus, we have obtained that when $\alpha = \gamma$ it is $\alpha \wedge (\beta \vee \alpha) = \alpha = \alpha \vee (\beta \wedge \alpha)$.

Let us now consider the strict ordering $\alpha < \gamma$. It is obvious that this condition implies that $\alpha = \{\underline{0}\}$ or $\gamma = \mathbb{R}^2$. But if $\alpha = \{\underline{0}\}$, then $\{\underline{0}\} \vee (\gamma \wedge \beta) = \gamma \wedge \beta = (\{\underline{0}\} \vee \gamma) \wedge \beta$, and if $\beta = \mathbb{R}^2$ it is $\alpha \vee (\gamma \wedge \mathbb{R}^2) = \alpha \vee \gamma = (\alpha \vee \gamma) \wedge \mathbb{R}^2$. ■

Lastly we have lattices, like the one shown in figure 1.5, which does not belong to the class of modular lattices. Indeed, for this lattice $b \leq a$ but $b \vee (c \wedge a) = b \vee 0 = b$ and $(b \vee c) \wedge a = 1 \wedge a = a$; and from $a \neq b$ we can deduce $b \vee (c \wedge a) \neq (b \vee c) \wedge a$.

1.4 Boolean lattices and algebras

The concept of complement has a great importance in lattice theory.

Definition 1.4.1 Let $\mathcal{L}$ be a bounded lattice. We identify the element $y \in \mathcal{L}$ as the *complement* of a given element $x \in \mathcal{L}$ if it is such that

$$x \vee y = 1 \quad \text{and} \quad x \wedge y = 0$$

It is easy to see that if y is the x complement, then x is the y complement as well. On the other hand the following relations $0 \vee 1 = 1$ and $0 \wedge 1 = 0$ imply that these two elements are complement of each other. Moreover it is easy to prove the following result

Proposition 1.4.2 *If in a distributive lattice $\mathcal{L}$ the complement of a given element exists, then it is unique.*

Proof – Suppose in fact that $\mathcal{L}$ is a distributive lattice and that a is an element with two complements a_1 and a_2. Then, from $a \vee a_2 = 1$ and $a \wedge a_1 = 0$ it follows that

$$a_1 = a_1 \wedge (a \vee a_2) = \text{distributivity} = (a_1 \wedge a) \vee (a_1 \wedge a_2) = a_1 \wedge a_2 \leq a_2 \,.$$

Similarly, $a_2 \leq a_1$, and so $a_1 = a_2$. ■

In a non distributive lattice this uniqueness property in general does not hold.

Example 1.4.3 In the modular lattice M5 of figure 1.6 both c and b are complements of the element a since $a \wedge c = a \wedge b = 0$ and $a \vee c = a \vee b = 1$. In general for any $x \in \{a, b, c\}$ the two elements $y \in \{a, b, c\}$ such that $y \neq x$ are both complements of x. ■

But there could exists a distributive bounded lattice in which only the elements 0 and 1 have the complement (which is 1 and 0, respectively).

Example 1.4.4 Let $\mathcal{L}$ be a totally ordered bounded lattice containing more than two elements. Then it is easy to see that this lattice is distributive, but no other element than 0 and 1 can have a complement. ■

Definition 1.4.5 We have a *complementable* bounded lattice when every element of the lattice has at least one complement.
We have a bounded *complemented* lattice when every element a of the lattice has just one complement, denoted by a'.

A *Boolean lattice* $\mathcal{B}$ is a distributive bounded lattice equipped with a *complementation* unary mapping, i.e., every element x has a well defined (unique) complement, denoted by x'. The operation of assigning the complement x' to every element x of the lattice defines a unary operation $'$ from $\mathcal{B}$ onto itself, called *complementation*. These very important structures were introduced by George Boole in 1854. At that time Boole was trying to define a logical calculus that could coincide with the propositional (enunciative) calculus. Later on it was possible to recognize as another important concrete Boolean lattice class, the power set $\mathcal{P}(X)$ of a *universe* X equipped with the set theoretic union and intersection, and the set theoretic complementation operations. Formally a boolean lattice is defined in the following way:

Definition 1.4.6 A *Boolean Lattice* is a structure $\langle \mathcal{B}, \wedge, \vee, ', 0, 1\rangle$ where

(b1) $\mathcal{B}$ is a set containing two distinct elements 0 and 1:

(b2) $\wedge$ and $\vee$ are two binary operations on $\mathcal{B}$ with respect to which the structure $\langle \mathcal{B}, \wedge, \vee, 0, 1\rangle$ is a distributive lattice bounded by the least element 0 and the greatest element 1: $\forall x \in \mathcal{B}$, $0 \leq x \leq 1$;

(b3) For every $x \in \mathcal{B}$ there exists $x' \in \mathcal{B}$ such that $x \wedge x' = 0$ and $x \vee x' = 1$; x' is the *complement* of x.

The following are very important examples of Boolean lattice:

Example 1.4.7 Let $\langle \{0, 1\}, \leq\rangle$ be the set consisting of the two elements $0, 1$ equipped with the usual order relation $0 \leq 1$. This poset is a Boolean lattice with respect to the operations presented in the tables below (at the left the lattice operations and at the right the complementation):

a	b	$a \wedge b$	$a \vee b$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

x	x'
0	1
1	0

This is usually known as the *two valued* or *two elements Boolean lattice*, denoted by $\mathbb{B} = \langle \{0, 1\}, \wedge, \vee, ', 0, 1\rangle$. ■

Example 1.4.8 The power set $P(X)$ of a universe X discussed in example 1.2.2 is a Boolean lattice if we choose the set theoretic complement $A^c = X \setminus A := \{x \in X : x \in X \text{ and } x \notin A\}$ as the complement of a given set A in the universe X. Such a Boolean lattice is $\mathbb{P} = \langle \mathcal{P}(X), \cap, \cup, {}^c, \emptyset, X \rangle$.

> "There is another interpretation of Boolean algebra in which the elements $a, b, c, \ldots$ stand for subsets of some "universal" set 1. In this interpretation $a \vee b$ means [...] union, and $a \wedge b$ the [...] common part of the sets a and b, and a' means the complement of a with respect to the set 1." [Fri38].

■

Example 1.4.9 The collection 2^X of all two valued functionals on the universe X introduced in example 1.2.4 is a Boolean lattice if we choose the functional $\chi' := 1 - \chi$ as the complement of a given functional χ. Such a lattice is $\mathbb{E} = \langle 2^X, \wedge, \vee, {}', \underline{0}, \underline{1} \rangle$. ■

Example 1.4.10 In example 1.2.11 we have investigated the lattice of all closed set of a topological space X. Taking inspiration from usual set theory, a natural way to introduce a complement of a generic closed set K could be its set complement K^c. The problem is that in general this latter is not a closed set. For instance, in the usual topology of $\mathbb{R}$ by closed sets the closed interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ (for $a \leq b$) has set complement $[a, b]^c = (-\infty, a) \cup (b, +\infty)$ which is not closed.

More generally, if $\mathcal{C}(X)$ is the collection of all closed sets (as primitive notion of a topology), the collection of all set complement of closed sets determines (as derived notion in this context) the family $\mathcal{O}(X)$ of all *open* sets: formally $O \in \mathcal{O}(X)$ iff there exists a closed set $K \in \mathcal{C}(X)$ such that $O = K^c$. On the basis of this definition and the properties (T1)–(T3) of example 1.2.10, it is easy to prove that the family $\mathcal{O}(X)$ satisfies the following conditions:

(O1) The empty set and the whole space X are open $\emptyset, X \in \mathcal{O}(X)$.

(O2) The set union of *any* family of open sets is open.

(O3) The *finite* intersection of open sets is open.

Now if one considers the collection $\mathcal{CO}(X)$ of all subsets of X which are both closed and open (the so–called *clopen* sets), then according to (T2) and (O2) the induced partial order structure is the one of complete lattice; moreover, for any clopen set U its set complement U^c is obviously clopen too and so one obtains that $\langle \mathcal{CO}(X), \cap, \cup, {}^c, \emptyset, X \rangle$ is a Boolean algebra.

In the topological space $\mathbb{R}$ the unique clopen sets are the trivial ones $\emptyset$ and $\mathbb{R}$, a very poor collection. A canonical way to obtain a sufficiently rich topology of clopen sets is the one at the basis of Pawlak rough sets, that will be considered in the forthcoming chapters. ■

Example 1.4.11 Let $D = \{1, 2, 5, 7, 10, 14, 35, 70\}$ be the set of all divisors of 70. Once defined $x \wedge y = \text{mcd}(x, y)$ ($5 \wedge 14 = 1$, $10 \wedge 35 = 5$), $x \vee y = \text{mcm}(x, y)$ ($5 \vee 14 = 70$, $10 \vee 35 = 70$) and $x' = \frac{70}{x}$ ($5' = 14$), the structure $\langle D, \wedge, \vee, ', 1, 70\rangle$ is a Boolean lattice. ■

Summarizing, a Boolean lattice is a distributive, complementable lattice. But we have seen that in a distributive lattice the uniqueness property of the complement of each element holds, and so in a complementable distributive lattice $\mathcal{B}$ we can introduce a unary operation $' : \mathcal{B} \mapsto \mathcal{B}$, $x \to x'$, the *complementation*, which associates with each element x its unique complement x'. In this way we can have the following equivalent, but equational, algebraic definition.

Definition 1.4.12 A *Boolean Algebra* is a sextuple $\langle \mathcal{B}, \wedge, \vee, ', 0, 1\rangle$, where:

(i) $\mathcal{B}$ is a set containing two distinct elements, 0 and 1,

(ii) $\wedge$ and $\vee$ are binary operations on $\mathcal{B}$,

(iii) $' : \mathcal{B} \mapsto \mathcal{B}$ is a unary operation from $\mathcal{B}$ onto $\mathcal{B}$,

such that for every $x, y, z \in \mathcal{B}$, the following axioms hold:

1) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$

2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

3) $x \vee 0 = x$ and $x \wedge 1 = x$

4) $x \vee x' = 1$ and $x \wedge x' = 0$

An alternative system of axioms for Boolean algebras is presented in [Bir67].

Definition 1.4.13 A structure $\langle \mathcal{B}, \wedge, \vee, ', 0, 1\rangle$ with $\wedge$ and $\vee$ binary operations on $\mathcal{B}$ and $'$ a unary operation on the same set, is a Boolean algebra iff the following properties are satisfied for all $a, b \in \mathcal{B}$:

(B1) $a \wedge b = (a' \vee b')'$

(B2) $a \vee b = b \vee a$

(B3) $a \vee (b \vee c) = (a \vee b) \vee c$

(B4) $(a \wedge b) \vee (a \wedge b') = a$

In this case axioms (B1)–(B4) are independent, that is no one of them can be derived from the other, so that eliminating one, the structure $\langle \mathcal{B}, \wedge, \vee, ', 0, 1\rangle$ is no more a Boolean algebra. Independence is an important property of an algebra, since enables us to determine which and only are the necessary properties to define it. Given a set of axioms, it is possible to prove the independence of one of them by finding a model (a concrete example) of the structure which satisfies all the axioms but this chosen one.

Example 1.4.14 Let us consider axioms (B1)–(B4), we show that (B2) is independent from the others. In order to do this we give a structure which satisfies (B1), (B3), (B4) and not (B2). The structure is the following: $\langle \{0,1\}, \wedge, \wedge, i\rangle$ (this means that $\vee := \wedge$), where $i(x) = x$ and $0 \wedge x = 0$, $1 \wedge x = 1$. As can be easily seen, (B2) is not satisfied: $0 \vee 1 = 0 \neq 1 = 1 \vee 0$. ■

Proposition 1.4.15 ([Bir67]). *In any Boolean algebra it is possible to prove that the following properties hold:*

(1) The lattice conditions

$a \vee a = a\,,\ a \wedge a = a$	*(idempotency)*
$a \vee b = b \vee a\,,\ a \wedge b = b \wedge a$	*(commutativity)*
$a \vee (b \vee c) = (a \vee b) \vee c$	*($\vee$ associativity)*
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$	*($\wedge$ associativity)*
$a \vee (a \wedge b) = a\,,\ a \wedge (a \vee b) = a$	*(absorption)*

(2) The distributivity conditions

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	*($\vee$ distributivity)*
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	*($\wedge$ distributivity)*

(3) The lattice zero and unit elements

$a \vee 0 = a\,,\ a \vee 1 = 1$	*($\vee$ boundary conditions)*
$a \wedge 0 = 0,\ a \wedge 1 = a$	*($\wedge$ boundary conditions)*

(4) The orthocomplementation conditions

$(a')' = a$	*(involution)*
$(a \wedge b)' = a' \vee b'$	*($\vee$ de Morgan law)*
$(a \vee b)' = a' \wedge b'$	*($\wedge$ de Morgan law)*
$a \vee a' = 1$	*(excluded middle law)*
$a \wedge a' = 0$	*(noncontradiction law)*

Also for the case of Boolean lattices, in [Sho51] it is possible to find a result which represents the analogous of theorem 1.2.14 (for lattices) and theorem 1.3.4 (for distributive lattices).

Theorem 1.4.16 *Let $\langle \mathcal{B}, \wedge, \vee\rangle$ be an algebraic structure where:*

1. *$\mathcal{L}$ is a non empty set,*

2. $\wedge$ and $\vee$ are two binary operations on $\mathcal{L}$ satisfying properties:

(DL1) $x = x \wedge (x \vee y)$;
(DL2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$;
(BL) *to each y there corresponds some y' such that*

$$x \wedge (y \vee y') = x \vee (y \wedge y')$$

Then, $\mathcal{L}$ is a Boolean lattice.

Now we can extend the notion of lattice isomorphism to the case of Boolean algebras.

Definition 1.4.17 Two Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be *isomorphic* iff there exists a lattice isomorphism $\phi : \mathcal{B}_1 \mapsto \mathcal{B}_2$ which preserves the complementation,

$$\forall x \in \mathcal{B}_1, \quad \phi(x') = \phi(x)'$$

Example 1.4.18 The lattice isomorphism seen in example 1.2.20 between the power set $\mathcal{P}(X)$ of a universe X and the collection 2^X of all the Boolean–valued functionals on X, $\chi : \mathcal{P}(X) \mapsto 2^X$, $A \in \mathcal{P}(X) \to \chi_A \in 2^X$, is indeed a Boolean lattice isomorphism since trivially for any subset A of X

$$\chi_{A^c} = 1 - \chi_A$$

■

As we have seen in example 1.4.8 the collection of all subsets of a universe X is a Boolean algebra under set inclusion (with associated set union and intersection) and set complementation. The following is an interesting *representation theorem* of finite Boolean algebras.

Theorem 1.4.19 *(Stone's Representation Theorem). Every finite Boolean algebra $\mathcal{B}$ is isomorphic to the algebra of parts of the set of its atoms.*

The preceding theorem causes the following important consequences:

(1) The cardinality of every finite Boolean algebra is a power of 2.

(2) Two finite Boolean algebras with the same cardinality are isomorphic to each other.

Let us explain this procedure with one example.

Example 1.4.20 In the figure 1.7 the left side is the Hasse diagram of the (finite) Boolean lattice B8, whose atoms are the elements a, b, and c. If one then considers the universe $X = \{a, b, c\}$ formed by all atoms of B8, the power set 2^3 of this universe ordered with the standard set theoretic ordering (the right side of the figure) is the Stone representation of B8.

The general procedure is the following one. Let $\mathcal{B}$ be a finite Boolean lattice and let us denote by $A(\mathcal{B})$ the collection of all its atoms, assumed as the universe

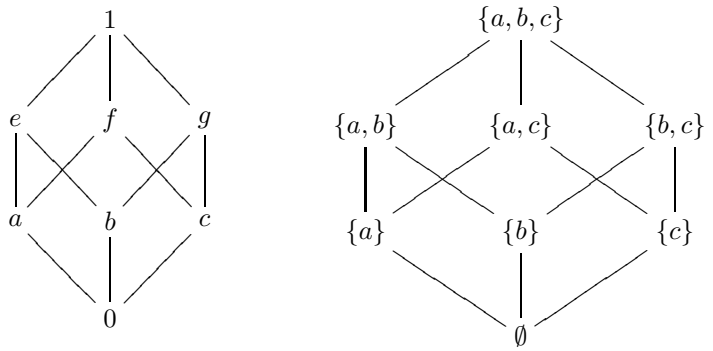

Figure 1.7: The finite Boolean lattice B8 with its Stone representation 2^3 of all subsets of the universe $\{a, b, n\}$.

of the Stone representation of $\mathcal{B}$. Then, let us construct the mapping $\phi : \mathcal{B} \mapsto \mathcal{P}(A(\mathcal{B}))$ which assigns to any element $x \in \mathcal{B}$ of the Boolean lattice the subset $\phi(x) = \{a \in A(\mathcal{B})) : a \leq x\}$, element of the power set $\mathcal{P}(A(\mathcal{B}))$ obtained as the collection of all atoms a which precede x. For instance in the example below, $\phi(e) = \{a, b\}$ since a and b are the two atoms which precede e. Trivially, the mapping ϕ is a Boolean lattice isomorphism. For instance, $\phi(e \vee g) = \phi(1) = \{a, b, c\} = \{a, b\} \cup \{b, c\} = \phi(e) \vee \phi(g)$. ■

The extension of this representation procedure to the case of an infinite Boolean lattice presents some problems since it is possible to give examples either of non–atomic or of non–complete Boolean algebras, and any Boolean algebra $\mathcal{P}(X)$ of all subsets of a universe X has both these properties. The best results about this situation are given by the following.

Proposition 1.4.21 *Let $\mathcal{B}$ be an abstract Boolean algebra, and let $A(\mathcal{B})$ denote the set of all its atoms. Then, the mapping $\phi : \mathcal{B} \mapsto \mathcal{P}(A(\mathcal{B}))$ associating with any element $x \in \mathcal{B}$ the subset of atoms $\phi(x) := \{a \in A(\mathcal{B}) : a \leq x\}$ is a homomorphism of Boolean lattices, i.e., both conditions* (1.7) *of definition 1.2.18 are satisfied by the mapping ϕ without the certainty that the bijective condition is assured. Moreover,*

(i) if $\mathcal{B}$ is atomic, then the mapping ϕ is injective;

(ii) if $\mathcal{B}$ is complete, the the mapping ϕ is surjective.

Proof – The fact that ϕ is a Boolean algebras homomorphism follows from the remark that an atom a satisfies the condition $a \leq x \vee y$ iff either $a \leq x$ or $a \leq y$. Analogously, an atom a satisfies the condition $a \leq x \wedge y$ iff both $a \leq x$ and $a \leq y$.

If $\mathcal{B}$ is atomic, then for any pair of its elements $x \neq y$ their symmetric difference $x \triangle y := (x \wedge y') \vee (y \wedge x')$ admits an atom $a \leq (x \triangle y)$; this implies that either

$a \leq x \wedge y'$ or $a \leq y \wedge x'$, but not in booth (otherwise for instance $a \leq x \wedge x' = 0$). Hence, a will be in just one, and only one, of $\phi(x)$ and $\phi(y)$.

If $\mathcal{B}$ is complete, then for any subset Y of $A(\mathcal{B})$ there exists the element $z = \bigvee\{y : y \in Y\}$ in $\mathcal{B}$ such that $\phi(z) = Y$. ■

As an immediate consequence we have.

Theorem 1.4.22 *A Boolean algebra $\mathcal{B}$ is isomorphic to the algebra $\mathcal{P}(X)$ of* all *subsets of some universe X iff it is atomic and complete.*

This result put the problem of what happens in all the other cases of Boolean algebras for which at least one of the above conditions does not hold. In example 1.4.10 we have seen that given a topological space X, the clopen subsets of X are closed under intersection, union and complementation forming a Boolean algebra. Note that in general the topology of clopen set $\mathcal{CO}(X)$ of a space X is contained in the power set of this latter, $\mathcal{CO}(X) \subseteq \mathcal{P}(X)$, in other words it is a sub-Boolean algebra of the power set. A well known theorem of Stone [Sto36] asserts that *every* Boolean algebra is isomorphically represented by one of such and that, if one requires the topological space to be compact Hausdorff as well as totally disconnected (what is now called *Stone space*, whereas Stone originally called these spaces with the name of *Boolean spaces*), this representation is essentially uniquely determined by the Boolean algebra. The meaning of this isomorphism is that any algebraic fact about Boolean algebras could be translated into a topological fact about Stone spaces, and vice versa.

1.4.1 Ortholattices

Point (4) of proposition 1.4.15 lists the conditions which are satisfied by the complement of a Boolean algebra. Relatively to this point, and as the consequence of the uniqueness of the complement a' of a given element a from the Boolean algebra $\mathcal{B}$, it is possible to introduce a unary operation $' : \mathcal{B} \mapsto \mathcal{B}$, $a \to a'$ which satisfies the condition of involution, both the de Morgan laws, the excluded middle and the noncontradiction laws. We can formalize this situation in the following way.

Definition 1.4.23 An orthocomplemented lattice, *ortholattice* for short, is a structure $\langle \mathcal{L}, \wedge, \vee, ', 0 \rangle$ of lower bounded lattice $\langle \mathcal{L}, \wedge, \vee, 0 \rangle$ equipped with an orthocomplementation mapping $' : \mathcal{L} \mapsto \mathcal{L}$ satisfying the equational conditions:

(oc1)	$x = x''$	(double negation)
(oc2a)	$x' \wedge y' = (x \vee y)'$	(de Morgan)
(oc3a)	$x \wedge x' = 0$	(noncontradiction)

With respect to the properties listed in point (4) of proposition 1.4.15, in the now introduced definition it is involved only one de Morgan property. Also the two properties of point (3) in the same proposition are simplified by the unique (oc3). But the following results hold.

Proposition 1.4.24 *In any lattice, under condition (oc1) the following are equivalent:*

(oc2a)	$x' \wedge y' = (x \vee y)'$	*(first de Morgan law)*
(oc2b)	$x' \vee y' = (x \wedge y)'$	*(second de Morgan law)*
(oc2c)	$x \leq y$ *implies* $y' \leq x'$	*(contraposition law)*
(oc2d)	$x' \leq y'$ *implies* $y \leq x$	*(dual contraposition law)*

In a similar way, in any bounded lattice under conditions (oc1) and (oc2) the following two are equivalent:

(oc3a)	$x \wedge x' = 0$	*(noncontradiction law)*
(oc3b)	$x \vee x' = 1$	*(excluded middle law)*

Let us stress that the just introduced notion of ortholattice can be applied to lattices which are not necessarily distributive, but with the lost of the uniqueness of the orthocomplementation mapping which characterizes Boolean algebras. In this way the complementation mapping assumes an important role, outside the "restricted" class of Boolean algebras. We clarify this fact with an example from the larger class of modular lattices.

Example 1.4.25 Let us consider the (6 elements) modular lattice M6 represented by the Hasse diagram of figure 1.8.

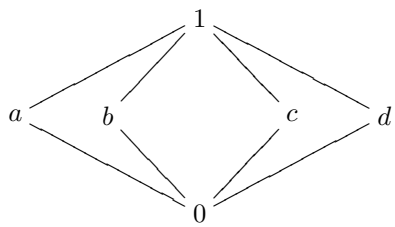

Figure 1.8: The 6 elements modular lattice M6

Then, any nontrivial element $x \neq 0, 1$ has 3 complements, precisely every nontrivial $y \neq 0, 1$ such that $y \neq x$. So this lattice is complementable, but it is possible to assign more than one orthocomplementation mappings, two of which are represented by the diagrams drawn in figure 1.9.

■

Example 1.4.26 Let us consider the two dimensional linear space $\mathbb{R}^2$. We have seen in example 1.2.11 that the collection $\mathcal{M}(\mathbb{R}^2)$ of all its subspaces is a modular (non distributive) lattice, bounded by the minimum subspace $\mathbf{0} = \{\underline{0}\}$ and the maximum subspace $\mathbf{1} = \mathbb{R}^2$. In this lattice any one–dimensional subspace α has a infinite collection of complements, i.e., any one–dimensional subspace $\beta \neq \alpha$

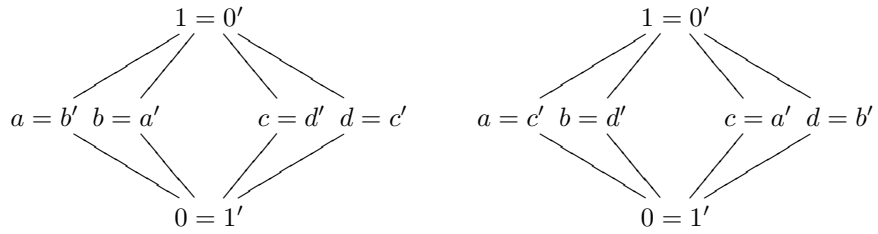

Figure 1.9: Two different orthocomplementations of the modular lattice M6 of figure 1.8.

is a complement of α. At a first glance it seems that there could be no intuitive criterium to select for each α one of these infinite subspaces, say α', as its *unique* orthocomplement.

But $\mathbb{R}^2$ can be equipped with the inner product defined for any pair of its vectors $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$ as the real number

$$\langle \underline{x} | \underline{y} \rangle := x_1\, y_1 + x_2\, y_2$$

The standard notion of *orthogonality* is the following one:

$$\underline{x} \perp \underline{y} \quad \text{iff} \quad \langle \underline{x} | \underline{y} \rangle = 0$$

Now, for any subspace M of $\mathbb{R}^2$ (i.e., the trivial subspaces $\{\underline{0}\}$ and $\mathbb{R}^2$, and all one dimensional subspaces) one can define its complement, denoted by $M^\perp$, as the subspace which is orthogonal to M. Formally,

$$M^\perp := \{\underline{h} \in \mathbb{R}^2 : \forall \underline{m} \in M,\ \langle \underline{h} | \underline{m} \rangle = 0\}$$

This is a subspace and the mapping $^\perp : \mathcal{M}(\mathbb{R}^2) \mapsto \mathcal{M}(\mathbb{R}^2)$, $M \to M^\perp$ is an orthocomplementation acting on a modular lattice. Probably, this example has been the cause of the term *ortho*complementation given to any mapping which satisfies the above conditions (oc1)–(oc3).

For instance, if one considers the subspace $M_{\pi/4} = \{(x, x) \in \mathbb{R}^2; x \in \mathbb{R}\}$ then its orthocomplement is $(M_{\pi/4})^\perp = \{(h, -h) \in \mathbb{R}^2 : h \in \mathbb{R}\} = M_{(3/4)\pi}$ since a fixed generic vector $(h_0, -h_0) \in (M_{\pi/4})^\perp$ is such that for any $(x, x) \in M_{\pi/4}$ the orthogonality condition $\langle (h_0, -h_0) | (x, x) \rangle = 0$ holds. On the other hand, $M_x = \{(x, 0) : x \in \mathbb{R}\}$ and $M_y = \{(0, y) : y \in \mathbb{R}\}$ are such that trivially $(M_x)^\perp = M_y$ and $(M_y)^\perp = M_x$. If one considers the modular sublattice of $\mathcal{M}(\mathbb{R}^2)$ consisting of the 6 subspaces $\{\{\underline{0}\}, M_x, M_{\pi/2}, M_y, M_{(3/4)\pi}, \mathbb{R}^2\}$, then the corresponding Hasse diagram has the form of the one depicted in the right side of figure 1.9 by the identifications $a = M_x$, $b = M_{\pi/2}$, $c = M_y$, and $d = M_{(3/4)\pi}$. ■

Let us make now a brief discussion about the semantical interpretation of the algebraic operations of an orthocomplemented lattice with respect to the ordinary propositional logic, the so–called *algebraic semantic* of a logic. First of all, let us stress that, "when interpreting [lattice] structures from the viewpoint of logic, it is customary to interpret the lattice elements $[a, b, \ldots]$ as *propositions*, the meet operation $[\wedge]$ as *conjunction* [logical AND], the join operation $[\vee]$ as *disjunction* [logical OR], and the orthocomplement $[\neg]$ as *negation* [logical NOT]." [Har81]. Hence, from the point of view of *algebraic semantic*, "clearly L1–L4 [of a lattice, according to theorem 1.2.12] are well-known properties of AND and OR in ordinary logic. This gives an algebraic reason for admitting as a *postulate* (if necessary) the statement that a given calculus of proposition is a lattice." [BvN36]. In this lattice context, "it is generally agreed that the partial ordering $\leq$ of a lattice [...] can be logically interpreted as a *relation* of implication, more specifically *semantic entailment*," which must not be confused with an implication logical *connective* which like conjunction and disjunction forms propositions out of propositions remaining at the same linguistic level.

In particular Boolean algebras have a significant role in classical logic.

> "**What is an algebra of logic?** Just what is meant by an "algebra of logic"? This question may be answered by giving a description of a Boolean algebra. Like the ordinary algebra of the real number system, a *Boolean algebra* consists of a set of elements, symbolized by the letters a, b, c, and so on, together with two *operations* or *rules of combination* called *addition* and *multiplication*, which assign to each pair of elements a and b of the algebra two further elements $a \vee b$ and $a \wedge b$ called the *sum* [disjunction] and *product* [conjunction] of a and b. There is also an operation called *negation*, which assigns to each element a an element a' called the *negative* of a.
> These operations satisfy certain postulates [...], which are in part the same postulates for ordinary algebra. Different algebras of logic, of course, will have different postulate systems.
> The interpretation of the Boolean elements and operations, however, is entirely different from that of ordinary algebra. The elements a, b, c, and so on, are to be thought of as representing not numbers, but *propositions* or *statements* or *sentences*." [Fri38].

Thus, once given the above interpretation of the elements of the algebra and of the connectives, all the postulates (axioms) of a Boolean algebra represents axioms of Boolean logic.

On the other hand, the modular distributive lattice of all subspaces of a finite dimensional (complex) linear space (or, better, the orthomodular version of any infinite dimensional complex separable Hilbert space) has a significant role in the so called quantum logic or logic of quantum mechanics, as described by Birkhoff and von Neumann in their seminal paper about this subject [BvN36]. Quoting from this latter: "The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion [...] is

that one can reasonably expected to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products* [i.e., intersection], *linear sums* [i.e., linear subspace generated by the set union], and *orthogonal complements* – and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*.

1.5 Propositional Boolean logic and crisp set semantic

We now turn our attention to *classical* (or *Boolean*) *propositional logic*. The objects at the basis of propositional logic are propositions, that is sentences that can be either true or false. True and false are called *truth values* and they are often identified respectively with the numbers 1 and 0, whose set has a standard structure of Boolean algebra:

$$\mathbb{B} = \langle \{0, 1\}, \wedge, \vee, -, 0, 1 \rangle$$

Let us remark that in this simplified introduction to classical propositional logic we identified a proposition a with its truth value. In particular, we consider the expression "$a = 1$" as a simplification of "the truth value of a is 1".

Making use of the algebraic operations $\wedge$, $\vee$, and $\neg$, considered as propositional *connectives*, propositions can be combined in different ways to obtain other propositions.

The negation of a proposition a, denoted as $\neg a$ and read as "NOT a", has truth value opposite to a, as can be seen in the following truth table 1.1.

a	$-a$
0	1
1	0

Table 1.1: NOT truth table.

The conjunction of two propositions a and b, denoted as $a \wedge b$ and read as "a AND b", is true when both propositions are true and false otherwise. The disjunction of two propositions p and q, denoted as $p \vee q$ and read as "p OR q", is true when one or both of p and q is true and false otherwise. These behaviors of conjunction and disjunction are described in table 1.2.

> "The sum $a \vee b$ is to be thought of as standing for the proposition that either a or b is true, and the product $a \wedge b$ is interpreted to mean the proposition that both a and b are true.
> The negation a' stands for the proposition that a is false. [...]
> All the algebras here discussed have the two special elements 0, standing for a proposition which is *certainly false*, and 1, representing a proposition which is *certainly true*." [Fri38].

a	b	$a \wedge b$	$a \vee b$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Table 1.2: AND and OR truth table.

Conjunction and disjunction are not independent: they can be defined one from the other using the negation connective. In fact, it is easily verified that the following property holds:

$$a \wedge b = -(-a \vee -b)$$

and vice versa

$$a \vee b = -(-a \wedge -b). \tag{1.8}$$

Remark 1.5.1 In this way the two valued Boolean algebra can also be denoted as a structure

$$\mathbb{B} = \langle \{0, 1\}, \vee, -, 0 \rangle$$

with the following derived notions defined as

$$a \wedge b := -(-a \vee -b)$$
$$1 := -0$$

in order to recover the original structure. This is in agreement with the comment: "An algebra of logic may also have other operations, but in Boolean algebra all operations are definable in terms of addition [disjunction] and negation." [Fri38].

Other important connectives in propositional logic are the *implication* and the *bi-implication* or equivalence. A proposition a "implies" a proposition b if a is true when b is true, so the truth value of "a IMPLIES b", denoted as $a \to b$, is 0 when a is true and b is false, and 1 otherwise.

a	b	$a \to b$
0	0	1
0	1	1
1	0	0
1	1	1

Table 1.3: Implication truth table.

Let us note that $a \to b$ has the same truth value of $-a \vee b$. Formally,

$$\forall a, b \in \{0, 1\} : \quad a \to b = -a \vee b \tag{1.9}$$

> "In Boolean logic, the operation of *implication* $a \to b$ (reads a implies b) is defined to be $-a \vee b$, that is "either a is false or b is true." It should be noted that implication is not a *relation* between elements like equality which either holds or does not hold, but a *rule of combination* like addition [disjunction], which produces from two elements of the algebra a third element." [Fri38].

Further, the negation connective can be obtained by the implication one according to

$$\forall a \in \{0,1\}: \quad -a := a \to 0 \tag{1.10}$$

Remark 1.5.2 For some generalizations which will be treated in the sequel, let us stress that the two valued Boolean algebra can be more economically denoted as a structure involving the only connective of implication and the constant value 0:

$$\mathbb{B} = \langle \{0,1\}, \to, 0 \rangle$$

Indeed, on the basis of this choice it is possible to define, as derived notions, the constant 1 and all the other connectives according to to following:

$$\begin{aligned} 1 &:= -0 \\ -a &:= a \to 0 \\ a \vee b &:= -a \to b = (a \to b) \to b \\ a \wedge b &:= -(a \to -b) = -(-a \vee -b) \end{aligned}$$

Let us quote from Łukasiewicz ([Bor70, p.87: "On three–valued logic"]):

> "Aristotelian logic, by assuming that every proposition is either true or false, distinguishes only two kinds of logical values, *truth* and *falsehood*. If truth is symbolized with 0 (...) and implication by $\to$, we can deduce all the laws of Aristotelian logic from the following principles and definitions:
>
> I. The *principles of implication*: $(0 \to 0) = (0 \to 1) = (1 \to 1) = 1$, $(1 \to 0) = 0$.
>
> II. The definitions of *negation*, *addition* [i.e., disjunction] and *multiplication* [i.e., conjunction]: $-a = (a \to 0)$, $(a \vee b) = [(a \to b) \to b]$, $a \wedge b = -(-a \vee -b)$.
>
> In these definitions, a and b are variables which may take on only two values, 0 or 1. All logical laws, expressed by means of variables, can be verified by the substitution of 0 and 1 for the letters."

Note that the following *minimal implication condition* (also called *boundary condition* in [SM87]) holds, relating the *implication connective* $\to$ to the *binary relation of ordering* $\leq$:

$$a \to b = 1 \quad \text{iff} \quad a \leq b$$

Two propositions a and b are equivalent, written as $a \leftrightarrow b$ or alternatively as "a IFF b", if $a \to b$ and $b \to a$. So the truth value of $a \leftrightarrow b$ is 1 both in the case "$a = 1$ and $b = 1$" and in the case "$b = 0$ and $a = 0$". That is,

$$a \leftrightarrow b = 1 \quad \text{iff} \quad a = b \tag{1.11}$$

From which it follows that

$$a \leftrightarrow b = 1 \quad \text{iff} \quad b \leftrightarrow a = 1 \tag{1.12}$$

a	b	$a \to b$	$b \to a$	$a \leftrightarrow b$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Table 1.4: Implication and bi-implication truth tables.

Exercise 1.5.3 Prove the following identities making us of the truth tables 1.4.

$$\forall a, b \in \{0, 1\} : \quad a \to (b \to a) = 1 \tag{1.13a}$$

$$\forall a, b, c \in \{0, 1\} : \quad (a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1 \tag{1.13b}$$

$$\forall a, b \in \{0, 1\} : \quad -a \to (a \to b) = 1 \tag{1.13c}$$

$$\forall a, b \in \{0, 1\} : \quad (a \to b) \to ((-a \to b) \to b) = 1 \tag{1.13d}$$

■

1.5.1 Classical propositional logic: a sketch

What we have discussed now is a naive picture of the link existing between a classical logical system and two–valued Boolean algebra. The complete formal approach is more complex and we are now going to briefly sketch it in a general context without any completeness claim.

First of all, we have to define an alphabet $\mathcal{A}$ and a set of formulas $\mathcal{F}$ on it. An *alphabet* $\mathcal{A} = \{V, L, U\}$ is a system composed by

- an infinite set V of *propositional variables* $(p, q, \ldots)$;
- a nonempty set of unary and binary *connectives* L (unary $\neg$, binary $\&$, o, and $\supset$);
- a set U of *auxiliary signs*, such as parentheses, punctuation, etc.

The union of the (assumed disjoint) sets V, L, U is the set of all *signs* of the alphabet $\mathcal{A}$.

Given an alphabet $\mathcal{A}$, the set of *well formed formulas* (wff) $\mathcal{F}$ is inductively defined by the following construction rules:

1. any propositional variable $p \in V$ is a wff;

2. if $\mathbf{p}$ and $\mathbf{q}$ are wff then $\neg\mathbf{p}$, $(\mathbf{p} \,\&\, \mathbf{q})$, $(\mathbf{p}\ o\ \mathbf{q})$, $(\mathbf{p} \supset \mathbf{q})$ are wff.

Now, the pair $\mathbf{L} = \langle \mathcal{A}, \mathcal{F} \rangle$ is the *formalized language* of classical propositional calculus. An *interpretation* (or *valuation*) is any mapping $v : V \mapsto \mathcal{T}$ which assigns to any propositional variable a truth value $t \in \mathcal{T}$ from the set $\mathcal{T}$ of *possible truth values*. An interpretation can be extended to the whole $\mathcal{F}$ in a straightforward manner, once the set $\mathcal{T}$ is equipped with a suitable structure of bounded lattice in which besides the lattice bounds $0, 1$ and the binary operations $\wedge$, $\vee$, a unary operation $-$ playing the role of lattice "negation" and a binary operation $\rightarrow$ playing the role of lattice "implication" connective are involved. To be precise, a valuation $v^* : \mathcal{F} \mapsto \mathcal{T}$ is a mapping which satisfies the conditions:

$$v^*(p) = v(p) \quad \text{if } p \in V \tag{1.14a}$$

$$v^*(\neg\mathbf{p}) = -v^*(\mathbf{p}) \tag{1.14b}$$

$$v^*(\mathbf{p} \,\&\, \mathbf{q}) = v^*(\mathbf{p}) \wedge v^*(\mathbf{q}) \tag{1.14c}$$

$$v^*(\mathbf{p}\ o\ \mathbf{q}) = v^*(\mathbf{p}) \vee v^*(\mathbf{q}) \tag{1.14d}$$

$$v^*(\mathbf{p} \supset \mathbf{q}) = v^*(\mathbf{p}) \rightarrow v^*(\mathbf{q}) \tag{1.14e}$$

As can be seen, the truth value of a compound formula depends only on the truth value of the elementary components (i.e., propositional variables) of the formula. This property is called *truth–functionality.*

Remark 1.5.4 In the classical logic case, in which the set of possible truth values is the Boolean one $\mathcal{T} = \{0, 1\}$, the primitive connectives can be more economically reduced, for instance, to the pair $\{o\ ,\ \neg\}$ since according to (1.8) and (1.9) it is possible to define $\mathbf{p} \,\&\, \mathbf{q} := \neg(\neg\mathbf{p}\ o\ \neg\mathbf{q})$ and $\mathbf{p} \supset \mathbf{q} := \neg\mathbf{p}\ o\ \mathbf{q}$.

This is not the case of other nonclassical logic systems in which, for instance, the implication connective could have a form different from $\neg\mathbf{p}\ o\ \mathbf{q}$ or in which, for instance, one of the de Morgan laws is rejected (see chapter **??**).

Remark 1.5.5 In order to facilitate the extension to the many–valued logical propositional calculus, another more economical way to introduce the formal language of classical propositional logic consists in adding to the language a constant symbol $\perp$ and taking as primitive the only binary connective of implication, i.e., the pair of operators $L = \langle \supset, \perp \rangle$.

Then, in order to obtain the other connectives as derived ones one introduces the definitions:

$$\begin{aligned} \neg\mathbf{p} &:= \mathbf{p} \supset \bot \\ \mathbf{p}\ o\ \mathbf{q} &:= \neg\mathbf{p} \supset \mathbf{q} \\ \mathbf{p}\ \&\ \mathbf{q} &:= \neg(\mathbf{p} \supset \neg\mathbf{q}) \end{aligned}$$

This is in agreement with remark 1.5.2 if one defines a Boolean interpretation as a mapping $v : V \to \{0,1\}$ such that its extension to wff $v^* : \mathcal{F} \to \{0,1\}$ satisfies the conditions:

$$\begin{aligned} & v^*(p) = v(p) \quad \text{if } p \in V \\ & v^*(\bot) = 0 \\ & v^*(\mathbf{p} \supset \mathbf{q}) = v^*(\mathbf{p}) \to v^*(\mathbf{q}) \end{aligned}$$

For instance, using these conditions one has that $v^*(\neg\mathbf{p}) = v^*(\mathbf{p} \supset \bot) = v^*(\mathbf{p}) \to v^*(\bot) = v^*(\mathbf{p}) \to 0$, with $v^*(\mathbf{p}) \in \{0,1\}$ whatever be $\mathbf{p}$, and so from (1.10) it follows that $v^*(\mathbf{p}) \to 0 = -v^*(\mathbf{p})$. Hence, $v^*(\neg\mathbf{p}) = -v^*(\mathbf{p})$.

All the other properties (1.14) of a valuation are proved in a similar way.

1.5.2 Boolean semantic of classical logic

In classical logic, of course, we have that the set of truth values is the two–valued set $\mathcal{T} = \{0,1\}$ with the associated Boolean algebra structure. A valuation is now a generic mapping $v : \mathcal{F} \to \{0,1\}$, in which we omit the index * for the sake of simplicity. The collection of all valuations is the set $\{0,1\}^{\mathcal{F}} = \{v_x : \mathcal{F} \mapsto \{0,1\} \mid x \in X\}$, where X denotes an index set which allows one to distinguish different valuations ($x \neq y$ in X implies $v_x \neq v_y$).

Definition 1.5.6 Let v be a valuation of the formalized language $\mathbf{L}$. If $\mathbf{p} \in \mathcal{F}$ is a wff and $v(\mathbf{p}) = 1$ then it is said that v *satisfies* $\mathbf{p}$, written $v \models \mathbf{p}$. In this case one also says that $\mathbf{p}$ is *true* with respect to the valuation v. Moreover, the formula $\mathbf{p}$ is said to be *false* with respect to v iff $v(\mathbf{p}) = 0$.

A formula is *satisfiable* if there exists at least one interpretation which satisfies it. A *tautology* is a formula which is true for all possible valuations and a *contradiction* a formula which is false for all valuations. We denote by $\models \mathbf{p}$ the fact that formula $\mathbf{p}$ is a tautology, i.e.,

$$\models \mathbf{p} \quad \text{iff} \quad \forall x \in X,\ v_x(\mathbf{p}) = 1$$

Moreover, for any finite set of formulas $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ we denote by $\models \mathbf{A}$ the fact that for every formula $\mathbf{a}_i \in \mathbf{A}$ it holds that $\models \mathbf{a}_i$ (i.e., $\forall \mathbf{a}_i \in \mathbf{A}$ and $\forall x \in X$: $v_x(\mathbf{a}_i) = 1$).

Remark 1.5.7 Using the properties (1.14) and taking into account the Boolean identities (1.13) it is trivial to obtain that for any valuation the

following identities hold:

$$\begin{aligned}
&\forall x \in X, \quad v_x(\mathbf{p} \supset (\mathbf{q} \supset \mathbf{p})) = 1\\
&\forall x \in X, \quad v_x((\mathbf{p} \supset (\mathbf{q} \supset \mathbf{r})) \supset ((\mathbf{p} \supset \mathbf{q}) \supset (\mathbf{p} \supset \mathbf{r}))) = 1\\
&\forall x \in X, \quad v_x(\neg\mathbf{p} \supset (\mathbf{p} \supset \mathbf{q})) = 1\\
&\forall x \in X, \quad v_x((\mathbf{p} \supset \mathbf{q}) \supset ((\neg\mathbf{p} \supset \mathbf{q}) \supset \mathbf{q})) = 1
\end{aligned}$$

That is the involved formulas are all tautologies.

A *deductive system* is defined as a structure $\langle \mathbf{L}, \mathfrak{A}, \{r_1, \ldots, r_k\}\rangle$ based on the formalized language $\mathbf{L}$, where $\mathfrak{A}$ is a set of tautologies, in this context called the *axioms* of the system, and $r_1, \ldots, r_k$ are *rules of inference*. In our classical case, $\mathfrak{A}$ contains the axioms characterizing Boolean logic:

(BL1) $\mathbf{p} \supset (\mathbf{q} \supset \mathbf{p})$

(BL2) $(\mathbf{p} \supset (\mathbf{q} \supset \mathbf{r})) \supset ((\mathbf{p} \supset \mathbf{q}) \supset (\mathbf{p} \supset \mathbf{r}))$

(BL3) $\neg\mathbf{p} \supset (\mathbf{p} \supset \mathbf{q})$

(BL4) $(\mathbf{p} \supset \mathbf{q}) \supset ((\neg\mathbf{p} \supset \mathbf{q}) \supset \mathbf{q})$

and the only inference rule is *modus ponens* : "if $\mathbf{p}$ and $\mathbf{p} \supset \mathbf{q}$ are true, than also $\mathbf{q}$ is true", written as $\mathbf{p}, \mathbf{p} \supset \mathbf{q} \vdash \mathbf{q}$ or also as:

$$\frac{\mathbf{p}, \mathbf{p} \supset \mathbf{q}}{\mathbf{q}}$$

As can be seen, this axiomatization takes into account only the connectives of implication and negation. Conjunction and disjunction can be obtained as: $\mathbf{p} \,\&\, \mathbf{q} := \neg(\mathbf{p} \supset \neg\mathbf{q})$ and $\mathbf{p} \mathbin{\mathfrak{o}} \mathbf{q} := (\mathbf{p} \supset \mathbf{q}) \supset \mathbf{q}$. This is not the case of other nonclassical logical systems in which, for instance, the implication connective can assume a different form than $\neg\mathbf{p} \mathbin{\mathfrak{o}} \mathbf{q}$ or in which de Morgan laws are not satisfied. In the sequel we will consider also the new connective

$$\mathbf{p} \Leftrightarrow \mathbf{q} := (\mathbf{p} \supset \mathbf{q}) \,\&\, \mathbf{q} \supset \mathbf{p}$$

Remark 1.5.8 From the semantical point of view, we have that for every valuation v

$$v(\mathbf{p} \Leftrightarrow \mathbf{q}) = 1 \quad \text{iff} \quad v(\mathbf{p}) = v(\mathbf{q})$$

Indeed, $v(\mathbf{p} \Leftrightarrow \mathbf{q}) = 1$ iff $1 = v((\mathbf{p} \supset \mathbf{q}) \,\&\, (\mathbf{q} \supset \mathbf{p})) = v(\mathbf{p} \supset \mathbf{q}) \wedge v(\mathbf{q} \supset \mathbf{p})$ and this holds when both $v(\mathbf{p} \supset \mathbf{q}) = 1$ and $v(\mathbf{q} \supset \mathbf{p}) = 1$. But from the table 1.4 it follows that this condition is satisfied when $v(\mathbf{p}) = v(\mathbf{q})$.

Given a deductive system a *deduction* of a proposition $\mathbf{q}$, starting from a set of propositions $\mathbf{A} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$, called *premises* is a sequence of propositions ending with $\mathbf{q}$ and obtained applying modus ponens to $\mathbf{p_i}$ and axioms. In this case we shall denote this fact by $\mathbf{p_1}, \ldots, \mathbf{p_n} \vdash \mathbf{q}$, or simply by $\mathbf{A} \vdash \mathbf{q}$.

Example 1.5.9 For instance, it is possible to deduce $\mathbf{p} \supset \mathbf{p}$ from the empty set of premises, in the following way:

$BL1$	$\mathbf{p} \supset ((\mathbf{p} \supset \mathbf{p}) \supset \mathbf{p})$
$BL2$	$\mathbf{p} \supset ((\mathbf{p} \supset \mathbf{p}) \supset \mathbf{p}) \supset ((\mathbf{p} \supset (\mathbf{p} \supset \mathbf{p})) \supset (\mathbf{p} \supset \mathbf{p}))$
MP	$((\mathbf{p} \supset (\mathbf{p} \supset \mathbf{p})) \supset (\mathbf{p} \supset \mathbf{p}))$
$BL1$	$(\mathbf{p} \supset (\mathbf{p} \supset \mathbf{p}))$
MP	$\mathbf{p} \supset \mathbf{p}$

■

If $\mathbf{A}$ is a finite set of wff and $\mathbf{p}$ a wff then $\mathbf{p}$ is a *logical consequence* of $\mathbf{A}$, written $\mathbf{A} \models \mathbf{p}$, iff $\forall x \in X, v_x \models \mathbf{A}$ implies $v_x \models \mathbf{p}$.

Two propositions are *logically equivalent* iff $\mathbf{p} \models \mathbf{q}$ and $\mathbf{q} \models \mathbf{p}$ and we write $\mathbf{p} \equiv \mathbf{q}$. Thus

$$\mathbf{p} \equiv \mathbf{q} \quad \text{iff} \quad \forall x \in X, \begin{cases} v_x(\mathbf{p}) = 1 \ \text{ implies } \ v_x(\mathbf{q}) = 1 \\ v_x(\mathbf{q}) = 1 \ \text{ implies } \ v_x(\mathbf{p}) = 1 \end{cases}$$

Remark 1.5.10 The above condition of logical equivalence of propositions can be summarized in the unique condition "$v_x(\mathbf{p}) = 1$ iff $v_x(\mathbf{q}) = 1$", which is equivalent to the condition "$v_x(\mathbf{p}) \neq 1$ iff $v_x(\mathbf{q}) \neq 1$", i.e., owing to the two valued semantic on $\{0, 1\}$, to "$v_x(\mathbf{p}) = 0$ iff $v_x(\mathbf{q}) = 0$". In other words, the condition of logical equivalence can be restated in the simpler form

$$\mathbf{p} \equiv \mathbf{q} \quad \text{iff} \quad \forall x \in X, \ v_x(\mathbf{p}) = v_x(\mathbf{q})$$

which, taking into account the remark 1.5.8, can be written as

$$\mathbf{p} \equiv \mathbf{q} \quad \text{iff} \quad \models (\mathbf{p} \Leftrightarrow \mathbf{q})$$

Examples of equivalent formulas are $\mathbf{p} \wedge \mathbf{q} \equiv \mathbf{q} \wedge \mathbf{p}$, $\mathbf{p} \vee \mathbf{q} \equiv \mathbf{q} \vee \mathbf{p}$, $\mathbf{p} \supset \mathbf{q} \equiv \neg\mathbf{q} \supset \neg\mathbf{p}$.

Of course $\equiv$ is an equivalence relation since it is reflexive, symmetric and transitive. Thus, given a formula $\mathbf{p} \in \mathcal{F}$ its equivalence class is defined as $[\mathbf{p}] := \{\mathbf{q} \in \mathcal{F} : \mathbf{p} \equiv \mathbf{q}\}$. The collection of all equivalence classes is $\mathcal{F}_{\equiv} := \{[\mathbf{p}] : \mathbf{p} \in \mathcal{F}\}$. On $\mathcal{F}_{\equiv}$ we define the following operators (which are well posed since they do not depend on the choice of the representative element selected in the involved equivalence classes):

$$\begin{aligned} -[\mathbf{p}] &:= [\neg\mathbf{p}] \\ [\mathbf{p}] \wedge [\mathbf{q}] &:= [\mathbf{p} \ \& \ \mathbf{q}] \\ [\mathbf{p}] \vee [\mathbf{q}] &:= [\mathbf{p} \ \mathfrak{o} \ \mathbf{q}] \end{aligned}$$

Further the equivalence class $\mathbf{1} := [\mathbf{p} \ \mathfrak{o} \ \neg\mathbf{p}]$ is the collection of all tautologies, and $\mathbf{0} := [\mathbf{p} \ \& \ \neg\mathbf{p}]$ is the collection of all contradictions.

Proposition 1.5.11 *The system* $\langle \mathcal{F}_{\equiv}, \wedge, \vee, -, \underline{0}, \underline{1} \rangle$ *is a Boolean algebra.*

Proof – The proof derives from the properties of Boolean logic. For instance, the commutativity of $\cdot$ is led by the commutativity of logical AND: $[\mathbf{p}] \wedge [\mathbf{q}] = [\mathbf{p} \,\&\, \mathbf{q}]$ and $[\mathbf{q}] \wedge [\mathbf{p}] = [\mathbf{q} \,\&\, \mathbf{p}]$. Since $\mathbf{p} \,\&\, \mathbf{q}$ and $\mathbf{q} \,\&\, \mathbf{p}$ are logically equivalent then, $[\mathbf{p}] \wedge [\mathbf{q}]=[\mathbf{q}] \wedge [\mathbf{p}]$. ■

This method of construction of an algebra from a logical system is called Lindenbaum–Tarski construction and the so obtained algebra the Lindenbaum–Tarski algebra [RS70]. Thus, the Lindenbaum–Tarski algebra of propositional logic is a Boolean algebra.

Of course, if we consider other logical systems than classical logic, we will obtain other structures than Boolean algebras.

The relationship between Boolean logic and Boolean algebras is described by the following proposition. The standard interpretation of a formula $\mathcal{I}_C^*$ is a mapping from formulas to the algebra $\mathbb{B}$ and the following proposition holds.

Proposition 1.5.12 *A formula* $\mathbf{p}$ *is provable in Boolean Logic BL, i.e.,* $\vdash \mathbf{p}$*, iff* $\mathbf{p}$ *is a tautology in* $\mathbb{B}$*, i.e, for all interpretations* v*,* $v \models \mathbf{p}$*.*

As it should now be clear, we are mainly dealing with "algebras of logic".

Let us now report an interesting quotation of M. Dunn which institutes a relationship from algebraic vs. set-theoretical semantic.

> It is conventional to think of a logical system has having two distinct aspects: *syntax* (well–formed strings of symbols, e.g., *sentences*) and *semantics* (what, e.g., these sentences mean, i.e., *propositions*). These double aspects compete with one another as can be seen in the competing usage 'sentential calculus' and 'propositional calculus', but we should keep firmly in mind both aspects.
>
> Since sentences can be combined by way of connectives, say the *conjunction* sign $\wedge$, to form further sentences, typically there is for each logical system at least one natural algebra arising at the level of syntax, the *algebra of sentences* (if one has a natural logical equivalence relation there is yet another that one obtains by identifying logically equivalent sentences together into equivalence classes – the so-called '*Lindenbaum algebra*'). And since propositions can be combined by the corresponding logical operations, say conjunction, to form propositions, there is an analogous algebra of proposition. [Dun86, p. 181].

1.5.3 Semantic of classical propositional logic based on crisp representation of sets

If X is a nonempty set, the universe of discourse, it is possible to define a set A through its *characteristic function* $\chi_A : X \mapsto \{0, 1\}$ which is expressed for all $x \in X$ by the law

$$\chi_A(x) := \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{otherwise} \end{cases}$$

The collection of all such characteristic functions, called *crisp* (or *sharp*, also *exact*) sets, will be denoted by $\mathcal{E}(X) := \{\chi_A \in \{0,1\}^X : A \in \mathcal{P}(X)\}$.

The lattice meet of any two characteristic functions $\chi_A, \chi_B \in \mathcal{E}(X)$ is the Boolean valued function $\chi_A \wedge \chi_B \in \mathcal{E}(X)$ defined $\forall x \in X$ in the following way:

$$(\chi_A \wedge \chi_B)(x) := \min\{\chi_A(x), \chi_B(x)\} = \begin{cases} 1 & \text{if} \quad x \in A \cap B \\ 0 & \text{otherwise} \end{cases} \tag{1.15}$$

which is the characteristic functional of the set theoretic intersection:

$$\chi_A \wedge \chi_B = \chi_{A \cap B}$$

Analogously, the corresponding join is given by

$$(\chi_A \vee \chi_B)(x) := \max\{\chi_A(x), \chi_B(x)\} = \begin{cases} 1 & \text{if} \quad x \in A \cup B \\ 0 & \text{otherwise} \end{cases} \tag{1.16}$$

which is the characteristic functional of the set theoretic union:

$$\chi_A \vee \chi_B = \chi_{A \cup B}$$

With respect to these binary operations 2^X turns out to be a distributive lattice bounded by the least element $\chi_\emptyset$ and the greatest element χ_X, explicitly given for arbitrary x running on X by

$$\chi_\emptyset(x) = 0 \quad \text{and} \quad \chi_X(x) = 1$$

In the sequel we will set $\underline{0} = \chi_\emptyset$ and $\underline{1} = \chi_X$ to denote the identical 0 and identical 1 functions on X, respectively.

The partial ordering relation induced from this lattice structure is the pointwise one

$$\chi_A \leq \chi_B \quad \text{iff} \chi_A \leq \chi_B \quad \text{iff} \forall x, \quad \chi_A(x) \leq \chi_B(x)$$

which corresponds to the set theoretic inclusion of the involved sets A and B:

$$\chi_A \leq \chi_B \quad \text{iff} \quad A \subseteq B$$

Finally, the complement of the characteristic function χ_A is the Boolean valued function

$$\chi'_A(x) := (\underline{1} - \chi_A)(x) = \begin{cases} 1 & \text{if} \quad x \in A^c \\ 0 & \text{otherwise} \end{cases} \tag{1.17}$$

Of course, this is the characteristic function of the set theoretic complement A^c of the set A:

$$\chi'_A = \chi_{A^c}$$

In this way we have obtained a Boolean algebra

$$\mathcal{E}(X) = \langle \{0,1\}^X, \wedge, \vee, ', \underline{0}, \underline{1} \rangle$$

which is isomorphic to the Boolean algebra

$$\mathbb{P} = \langle \mathcal{P}(X), \cap, \cup, {}^c, \emptyset, X \rangle$$

by the one-to-one and onto Boolean algebras morphism:

$$\chi : \mathcal{P}(X) \mapsto \{0,1\}^X, \quad A \to \chi_A \tag{1.18}$$

If we accept the paradigmatic approach to semantic in which propositions are represented by a set of 'possible worlds' (see [Dun86]), then we can formalize the relationship between sentences of the logical language and propositions as a mapping

$$A : \mathcal{F} \mapsto \mathcal{P}(X), \quad \mathbf{p} \to A(\mathbf{p}) \tag{1.19}$$

which associates with any sentence $\mathbf{p}$ the set of 'possible worlds' ('situations', 'set–ups', 'states') $A(\mathbf{p})$ in which the sentence is true, i.e., $x \in A(\mathbf{p})$ iff $x \models \mathbf{p}$ (in other words, iff $v_x(\mathbf{p}) = 1$), under the conditions:

$$A(\mathbf{p} \,\&\, \mathbf{q}) = A(\mathbf{p}) \cap A(\mathbf{q}) \tag{1.20a}$$

$$A(\mathbf{p} \circ \mathbf{q}) = A(\mathbf{p}) \cup A(\mathbf{q}) \tag{1.20b}$$

$$A(\neg \mathbf{p}) = A(\mathbf{p})^c \tag{1.20c}$$

Making use of the (1.18) it is possible to make a further extension of (1.19) obtaining the following mapping $\Phi : \mathcal{F} \mapsto \{0,1\}^X$:

$$\mathbf{p} \in \mathcal{F} \xrightarrow{A} A(\mathbf{p}) \in \mathcal{P}(X) \xrightarrow{\chi} \chi_{A(\mathbf{p})} \in \{0,1\}^X$$

from which it can be induced also the so-called *valuation mapping*:

$$v : X \times \mathcal{F} \mapsto \{0,1\}, \quad (x, \mathbf{p}) \to v(x, \mathbf{p}) := \chi_{A(\mathbf{p})}(x)$$

which associates with a semantical world $x \in X$ and a sentence $\mathbf{p}$ its truth value $\chi_{A(\mathbf{p})}(x) \in \{0,1\}$, the sentence $\mathbf{p}$ is 'true' (resp., 'false') in the world x iff its truth value is 1 (resp., 0), i.e., $x \in A(\mathbf{p})$ (resp., $x \notin A(\mathbf{p})$). Of course, the above (1.20) become:

$$\begin{aligned} \Phi(\mathbf{p} \,\&\, \mathbf{q}) &= \chi_{A(\mathbf{p})} \wedge \chi_{A(\mathbf{q})} \\ \Phi(\mathbf{p} \circ \mathbf{q}) &= \chi_{A(\mathbf{p})} \vee \chi_{A(\mathbf{q})} \\ \Phi(\neg \mathbf{p}) &= \left(\chi_{A(\mathbf{p})}\right)^c \end{aligned}$$

For any fixed semantical world $x \in X$ one has the *valuation* induced from x as the mapping

$$v_x : \mathcal{F} \mapsto \{0,1\}, \quad \mathbf{p} \to v_x(\mathbf{p}) = \chi_{A(\mathbf{p})}(x)$$

which in the fixed semantical world x associates with any sentence $\mathbf{p}$ its truth value $v_x(\mathbf{p}) \in \{0,1\}$. Let us note that the collection of all valuation can be denoted by

$$\{0,1\}^{\mathcal{F}} := \{v_{\hat{x}} : \mathcal{F} \mapsto \{0,1\} \mid \hat{x} \in \widehat{X}\}$$

where $\widehat{X}$ is an index set, i.e., the collection of all indices which allows one to distinguish different valuations: $\hat{x} \neq \hat{y}$ implies $v_{\hat{x}} \neq v_{\hat{y}}$.

If this is the general setup of the semantic aspect of a formalized language, let us adopt the following

Semantical Assumption: the collection X of all possible semantical worlds coincides with the set $\widehat{X}$ of all indices which distinguish different valuations.

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